A Generalized Polynomial Set

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Introduction

In this paper we establish certain formulas giving generalized polynomial set.

1. A General class of Polynomials

The polynomials such as the Hermite polynomials, the Jacobi polynomials (and its special cases which bear the names Gegenbauer, the Bessel polynomials, the Gould-Hopper Polynomials, the Konhauser biorthogonal polynomials and several other polynomials play an important role in the study of the applied mathematics and physics. Almost all the above-mentioned polynomials are the special cases of a general class of polynomials introduced by Shrivastava**:**

$$S^{\alpha,\ \beta,\ \tau}_{n}[x] = \sum_{j=0}^{[\frac{N}{M}]} \frac{(-N)_{Mj}}{j!} A_{N,\ j}\, x^{j}, \qquad \text{... (A.1)}$$

$N = 0, 1, 2, \dots \dots,$

Where M is arbitrary positive integer and the coefficients $A_{N,\ j}\,(N, j\ \geq 0)$ are arbitrary constants, real or complex. Further, a multivariable analogue of the polynomial (1) has also been introduced by Shrivastava and Garg.

2. A Generalized polynomial Set

In an attempt to unify classical polynomial of the mathematical physics, Dhilon ,S.S. defined and studied a generalized function namely $z^{(\alpha,\ \beta,\ \tau)}_{n}[x; r; s;\ q, A, B, m. k. l\,]$.

Again, in a similar attempt of unifying various classical polynomial of mathematical physics, Joshi and Prajapat considered the following operator:

$$T_{k,\ l} = x^{q}(k + x\frac{d}{dx}), \qquad \text{.... (A.2)}$$

Where k is a constant and introduced the polynomial set $\{M^{\alpha}_{n}(x; r, p, b, k, q)\ ; n = 0, 1, 2, \dots \dots\}$ defined by

$$M^{\alpha}_{n}(x; r, p, b, k, q) = c(b, n)\, x^{-\alpha-qn-n}\, e^{p\,x^{r}} M^{\ n}_{k,q}(x^{\alpha+qn}\ e^{-p\,x^{r}}), \qquad \text{... (A.3)}$$

Where $c(b, n)$ is a constant, b, being a non-negative integer.

Clearly (3) is the generalization of the generalized Hermite polynomials of Gould and Hopper and generalized Laguerre polynomials. Motivated by above mentioned generalizations, Raizada recently introduced and studied a generalized polynomial set, which is defined as follows.

2.1 The Generalized Polynomial Set

Raizada has recently introduced and studied a generalized polynomial set that is defined by the following Rodrigues-type formulas [5, p. 64, (2.1.8)]:

$$S_{n}^{\alpha,\ \beta,\ \tau}[x;r,s,q,A,B,m.k.l\,] = (Ax+B)^{-\alpha}(1-\tau x^r)^{-\frac{\beta}{\tau}}\,T_{k,\ l}^{\,m+n}\,[\,(Ax+B)^{-\alpha+qn}(1-\tau x^r)^{-\frac{\beta}{\tau}+sn}], \qquad \text{... (B.1)}$$

Where the differential operator $T_{k,\ l}$, is defined as

$$T_{k,\ l} = x^l\left(k + x\frac{d}{dx}\right), \qquad \text{..... (B.2)}$$

The explicit form of this generalized polynomial set [5, p. 71, (2.3.4)] is:

$$S_{n}^{\alpha,\ \beta,\ \tau}[x;r,s,q,A,B,m.k.l] = B^{qn}\,x^{l(m+n)}(1-\tau x^r)^{sn}\sum_{p=0}^{m+n}\ \sum_{e=0}^{p}\ \sum_{\delta=0}^{m+n}\ \sum_{e=0}^{\delta}\ \frac{(-1)^{\delta}\ (-p\,)_e\ (\delta)_i\,(\alpha)_\delta}{p!\ \delta!\ \ i!\ \ e!}\ \frac{(\alpha-qn)_i}{(1-\alpha-\delta)_i}\left(-\frac{\beta}{\tau}-sn\right)_p\left(\frac{i+k+re}{l}\right)_{m+n}\left(\frac{\tau x^r}{1-\tau x^r}\right)^p\left(\frac{Ax}{B}\right)^{\delta}, \qquad \text{..... (B.3)}$$

The polynomial set defined by (B.1) is quite general in nature, and it unifies several classical polynomials introduced and studied by various research workers such as Chatterjea [1], Dhillon [2], Gould and Hopper [3], Krall and Frink [4], Singh N and Shrivastava [6], etc. some of the special cases of this polynomial set is given by Raizada [5, p. 65] in a tabular form. We mention below the following important special cases of (B.1):

(i) Taking $A = 1,\ B = 0$, in (B.1) and proceeding on lines similar to those mentioned by Raizada [5, p. 68-71] for obtaining the explicit form (b.3), we find that

$$S_{n}^{\alpha,\ \beta,\ \tau}[x;r,s,q,l,o,m,k,l] = (1-\tau x^r)^{sn}\ l^{m+n}\ x^{qn+l(m+n)}$$

$$\sum_{p=0}^{m+n}\ \sum_{e=0}^{p}\ \frac{(-p)_e}{p!\ e!}\left(\frac{\alpha+qn+k+re}{l}\right)_{m+n}\ \left(-\frac{\beta}{\tau}-sn\right)_p\ \left(\frac{\tau x^r}{1-\tau x^r}\right)^p, \qquad \text{..... (B.4)}$$

(ii) Further letting $\tau \to 0$, in (B.4) and using the following result

$$\mathrm{Lim}_{|b|\to\infty}(b)_n\left(-\frac{z}{b}\right)^n = z^n,$$

therein, we arrive at the following polynomial set:

$\lim_{\tau \to 0} S_{n}^{\alpha,\ \beta,\ \tau}[x; r, s, q, l, o, m, k, l] = S_{n}^{\alpha,\ \beta,\ 0}[x; r, s, q,$
$l, o, m, k, l] = x^{qn+l(m+n)}\ l^{m+n} \sum_{p=0}^{m+n} \sum_{e=0}^{p} \frac{(-p)_e}{p!\ e!} \left(\frac{\alpha+qn+k+re}{l}\right)_{m+n} (\beta x^r)^p,$

.... (B.5)

(iii) If we take $q = k = m = 0, l = -1$, in (B.5), then

$S_{n}^{\alpha,\ \beta,\ 0}[x; r, 0, 1, 0, 0, 0, -1] = (-1)^n H_{n}^{(r)}(x, \alpha,\ \beta) = (-x)^{-n} \sum_{p=0}^{n} \sum_{e=0}^{p} \frac{(-p)_e}{p!\ e!}(-\alpha -$
$re)_n (\beta x^r)^p$, ... (B.6)

Where $H_{n}^{(r)}(x, \alpha,\ \beta)$ is the class of the polynomial studied by Gould and Hopper [3].

(iv) Again, taking $q = 1,\ m = k = 0, l = -1$, in (B.5), we get

$S_{n}^{\alpha,\ \beta,\ 0}[x; r, 1, 1, 0, 0, 0, -1] = n!\ L_{n}^{\alpha}(x, \alpha, \beta)$

$= (-1)^{-n} \sum_{p=0}^{n} \sum_{e=0}^{p} \frac{(-p)_e}{p!\ e!}(-\alpha - n - re)_n (\beta x^r)^p$, ... (B.7)

Where $L_{n}^{\alpha}(x, \alpha,\ \beta)$ is known as the Singh and Shrivastava [6] polynomial.

(v) Also, if we take $m = k = 0, l = -1$, in (B.5), we arrive at the following relationship:

$S_{n}^{\alpha,\ \beta,\ 0}[x; r, q, 1, 0, 0, 0, -1] = F_{n}^{(r)}(x, \alpha, q, \beta)$

$= (-x)^{-q-1} \sum_{p=0}^{n} \sum_{e=0}^{p} \frac{(-p)_e}{p!\ e!}(-\alpha - qn - re)_n (\beta x^r)^p$, ... (B.8)

Where $F_{n}^{(r)}(x, \alpha, q, \beta)$ is known as the Chatterjea [1] polynomial.

(vi) Similarly, if we take $l = \tau = -1,\ q = 2, m = k = 0, \alpha = a - 2, \beta = b$, in (B.5), we get

$S_{n}^{(a-2,\ b,\ 0)}[x; -1,2, 1, 0, 0, 0, -1] = b^n\ Y_n(x, a, b)$

$= (-x)^{-n} \sum_{p=0}^{n} \sum_{e=0}^{p} \frac{(-p)_e}{p!\ e!}(-\alpha - 2 - 2n - e)_n \left(\frac{b}{x}\right)^p$, ... (B.9)

Where $Y_n(x, a, b)$ is known as the Krall and Frink [4] polynomial.

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References

1. Chatterjea, S. K., 'Quelques functions generatices des polynomes d' Hermite, du point de vue de l'algebre de lie', C. R. Acad. Sci. Paris Ser. A-B. vol. 268 (1969), P. A600-A602.

2. Dhillon, S. S., 'A study of generalization of special functions of mathematical Physics and their applications; Ph. D. Thesis', Bundelkhand University, Jhansi, India, (1989).

3. Gould, H. W. and Hopper, A. T.., 'Operational formulas connected with two generalizations of Hermite polynomials', Duke Math. J., vol. 29 (1962), p. 51-63.

4. Krall H. L. and Frink O., 'A new class of orthogonal polynomials, the Bessel polynomials,' Trans. Amer. Math. Soc. vol. 65 (1949), p. 110-115.

5. Raizada, S. K. 'A study of unified representation of special functions of Mathematical Physics and their use in statistical and boundary value problems', Ph.D. Thesis, Bundelkhand University, Jhansi, India (1191).

6. Singh, R. P. and Srivastava K. N., 'A note on generalization of Laguerre and Humbert polynomials', Ricerca (Napoli) (2), vol. 14 (1963), Settembre-Decembre, p. 11-21, Errata, Ibid (20 vol. 15 (1964) Maggio-Agosto, 63.