Investigating Special Function Relationships within Cantor Sets and Special Integral Transformations Using Local Fractional Operators

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Abstract

Investigated are the mappings for a few unique functions on Cantor sets. Three local fractional differential equations were then solved using the local fractional Fourier series, Fourier transforms, and Laplace transforms, and the related nondifferentiable solutions were provided.

Keywords: Fractional Operators, Stochastic Linear Volterra Equations, Fractional Nonlinear Systems

Introduction

Special functions play a significant role in mathematical analysis, function analysis, physics, and various other fields. Notable examples include the Gamma function, hypergeometric function, Bessel functions, Whittaker function, G-function, q-special functions, Fox's H-functions, Mittag-Leffler function, and Wright's function.

The Mittag-Leffler function has proven to be valuable in addressing practical problems. It has been applied to various scenarios, such as solving fractional evolution processes and providing solutions for fractional reaction-diffusion equations. Additionally, it has been used to establish the stability of fractional order nonlinear dynamic systems and model anomalous relaxation in dielectrics. Applications in continuous-time finance and fractional radial diffusion in a cylinder have also been explored, along with the Mittag-Leffler stability theorem for fractional nonlinear systems with delay. Stochastic linear Volterra equations of convolution type have been formulated based on the Mittag-Leffler function.

Recently, a novel approach utilizing Mittag-Leffler functions on Cantor sets, employing fractal measures, led to the development of special integral transforms grounded in local fractional calculus theory. This work explores applications of the local fractional calculus theory.

The primary objective of this paper is to investigate mappings associated with special functions defined on Cantor sets and to explore the practical applications of special integral transforms for addressing problems that lack differentiability.

The paper's structure is as follows:

Section 2 delves into the investigation of mappings for special functions defined on Cantor sets.

Section 3 covers special integral transforms within the context of local fractional calculus theory and their applications in addressing problems with nondifferentiable characteristics.

Section 4, the paper concludes with a summary of key findings and implications.

2. Mappings for Special Functions on Cantor Sets

In order to give the mappings for special functions on Cantor sets, we first recall some basic definitions about the fractal measure theory. Let Lebesgue-Cantor staircase function be defined as

$$H_\alpha(F \cap (0,x)) = \Gamma(1+\alpha)_0 I_x^{(\alpha)} 1, \quad (1)$$

where F is a cantor set, $H_\alpha(\cdot)$ is the α-dimensional Hausdorff measure, ${}_0I_x^{(\alpha)}(\cdot)$ is local fractional integral operator [24-31], and $\Gamma(\cdot)$ is a Gamma function. Following (1), we obtain

$H_\alpha(F \cap (0,x)) = x^\alpha,$ (2)

which is a Lebesgue-Cantor staircase function. For its graph

In this way, we define some real-valued functions on Cantor sets as follows. The Cantor staircase function is defined as

$f(x) = x^{2\alpha},$ (3)

And its graph is shown in Figure 1. The Mittag-Leffler functions on Cantor sets are given by

$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1+k\alpha)},$ (4)

and we draw the corresponding graph in Figure 2.

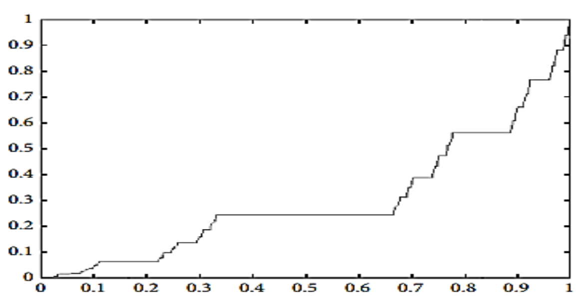

FIGURE 1: Graph of $x^{2\alpha}$ for $\alpha = \ln 2/\ln 3$.

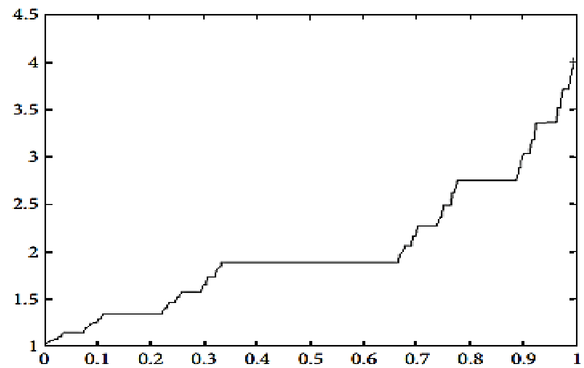

FIGURE 2: Graph of $E_\alpha(x^\alpha)$ for $\alpha = \ln 2/\ln 3$.

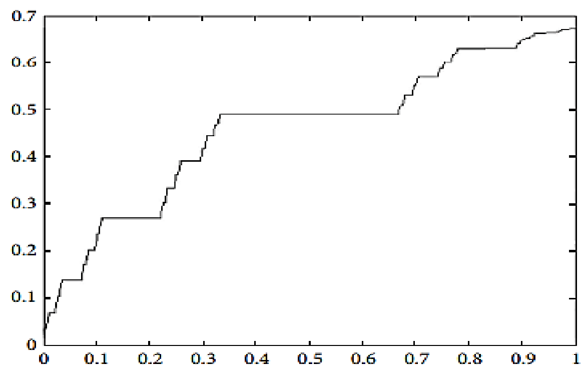

FIGURE 3: Graph of $\sin_\alpha x^\alpha$ for $\alpha = \ln 2/\ln 3$.

The sine on Cantor sets is defined by

$$\sin_\alpha x^\alpha = \sum_{k=0}^{\infty} (-1)^k \frac{x^{\alpha(2k+1)}}{\Gamma[1+\alpha(2k+1)]}, \quad (5)$$

and its corresponding graph is depicted in Figure 3. The cosine on Cantor sets is

$$\cos_\alpha x^\alpha = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2\alpha k}}{\Gamma(1+2\alpha k)}, \quad (6)$$

with graph in Figure 4.
Hyperbolic sine on Cantor sets is defined by

$$\sinh_\alpha x^\alpha = \sum_{k=0}^{\infty} \frac{x^{\alpha(2k+1)}}{\Gamma[1+\alpha(2k+1)]}, (7)$$

and we draw its graphs as shown in Figure 5.

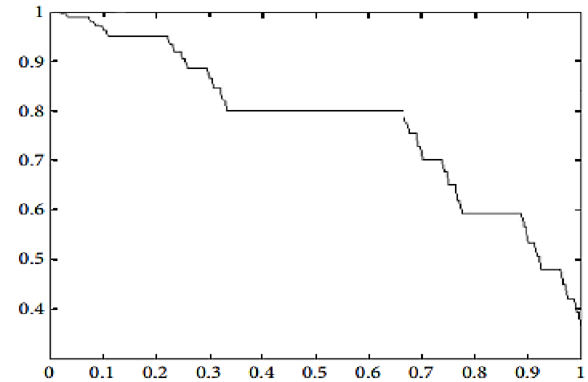

FIGURE 4: Graph of $\cos_\alpha x^\alpha$ for $\alpha = \ln 2/\ln 3$.

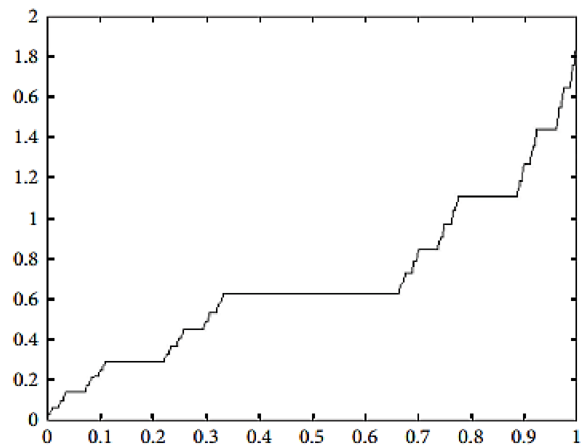

FIGURE 5: Graph of $\sinh_\alpha x^\alpha$ for $\alpha = \ln 2/\ln 3$.

Hyperbolic cosine on Cantor sets is defined as

$\cosh_{\alpha} x^{\alpha} = \sum_{k=0}^{\infty} \frac{x^{2\alpha k}}{\Gamma(1+2\alpha k)}$, (8)

and its graph is shown in Figure 6. Following (4)-(8), we have

$E_{\alpha}(i^{\alpha} x^{\alpha}) = \cos_{\alpha} x^{\alpha} + i^{\alpha} \sin_{\alpha} x^{\alpha}$, (9)

where i^{α} is a fractal unit of an imaginary number. If for $\varepsilon, \delta > 0$ and $\varepsilon, \delta \in R$, $f(x)$ satisfies the condition

$|f(x) - f(x_0)| < \varepsilon^{\alpha}$; (10)

for $x \in [a, b]$ we write it as follows:

$f(x) \in C_{\alpha}(a, b)$. (11)

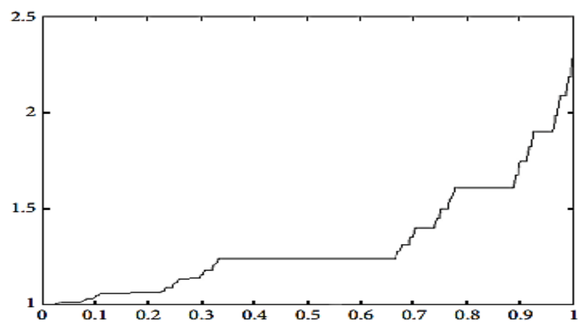

Figure 6: Graph of $\cosh_{\alpha} x^{\alpha}$ for $\alpha = \ln 2/\ln 3$.

3. Special Integral Transforms within Local Fractional Calculus

In this section, we introduce the conceptions of special integral transforms within the local fractional calculus concluding the local fractional Fourier series and Fourier and Laplace transforms. After that, we present three illustrative examples.

3.1. Definitions of Special Integral Transforms within Local Fractional Calculus. We here present briefly some results used in the rest of the paper.

Let $f(x) \in C_{\alpha}(-\infty, \infty)$. Local fractional trigonometric Fourier series of $f(x)$

$$f(x) = a_0 + \sum_{i=1}^{\infty} a_k \sin_{\alpha}(k^{\alpha} \omega_0^{\alpha} x^{\alpha}) + \sum_{i=1}^{\infty} b_k \cos_{\alpha}(k^{\alpha} \omega_0^{\alpha} x^{\alpha}). \quad (12)$$

The local fractional Fourier coefficients read as

$$a_0 = \frac{1}{T^\alpha}\int_0^T f(x)(dx)^\alpha,$$

$$a_k = \left(\frac{2}{T}\right)^\alpha \int_0^T f(x)\sin_\alpha(k^\alpha \omega_0^\alpha x^\alpha)(dx)^\alpha, \quad (13)$$

$$b_k = \left(\frac{2}{T}\right)^\alpha \int_0^T f(x)\cos_\alpha(k^\alpha \omega_0^\alpha x^\alpha)(dx)^\alpha.$$

We notice that the above results are obtained from Pythagorean theorem in the generalized Hilbert space.

Let $f(x) \in C_\alpha(-\infty,\infty)$. The local fractional Fourier transform of $f(x)$

$$\begin{aligned} F_\alpha\{f(x)\} &= f_\omega^{F,\alpha}(\omega) \\ &= \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty} E_\alpha(-i^\alpha\omega^\alpha x^\alpha) f(x)(dx)^\alpha \end{aligned} \quad (14)$$

The inverse formula is expressed as follows:

$$\begin{aligned} f(x) &= F_\alpha^{-1}(f_\omega^{F\alpha}(\omega)) \\ &= \frac{1}{(2\pi)^\alpha}\int_{-\infty}^{\infty} E_\alpha(i^\alpha\omega^\alpha x^\alpha) f_\omega^{F\alpha}(\omega)(d\omega)^\alpha. \end{aligned} \quad (15)$$

Let $f(x) \in C_\alpha(-\infty,\infty)$. The local fractional Laplace transform of $f(x)$ is defined as [24,32,33]

$$\begin{aligned} L_\alpha\{f(x)\} &= f_s^{L,\alpha x}(s) \\ &= \frac{1}{\Gamma(1+\alpha)}\int_0^{\infty} E_\alpha(-s^\alpha x^\alpha) f(x)(dx)^\alpha. \end{aligned} \quad (16)$$

The inverse formula local fractional Laplace transform of $f(x)$ is derived as [24,32,33]

$$\begin{aligned} f(x) &= L_\alpha^{-1}\{f_s^{L,\alpha}(s)\} \\ &= \frac{1}{(2\pi)^\alpha}\int_{\beta-i\infty}^{\beta+i\infty} E_\alpha(s^\alpha x^\alpha) f_s^{L,\alpha}(s)(ds)^\alpha, \end{aligned} \quad (17)$$

where $f(x)$ is local fractional continuous, $s^\alpha = \beta^\alpha + i^\alpha\infty^\alpha$, and $\mathrm{Re}(s) = \beta > 0$.

For more details of special integral transforms via local fractional calculus, see and the references therein.

3.2. Applications of Local Fractional Fourier Series and Fourier and Laplace Transforms to the Differential Equation on Cantor Sets. We now present the powerful tool of the methods presented above in three illustrative examples.

Example 1. Let us begin with the local fractional differential equation on Cantor set in the following form:

$a\frac{d^{\alpha}y}{d^{\alpha}x}+$ by $(x)=f(x),\ x\in(-\infty,+\infty)$, (18)

where a and b are constants and the nondifferentiable function $f(x)$ is periodic of period 2π so that it can be expanded in a local fractional Fourier series as follows:

$f(x)=\sum_{n=1}^{\infty}\ \sin_{\alpha}(n^{\alpha}x^{\alpha})$. (19)

Here, we give a particular solution in the following form:

$$y_p(x)=\begin{array}{l} a_0+\sum_{n=1}^{\infty}\ A_n\sin_{\alpha}(n^{\alpha}x^{\alpha}) \\ +\sum_{n=1}^{\infty}\ B_n\cos_{\alpha}(n^{\alpha}x^{\alpha}). \end{array} \quad (20)$$

Following (20), we have

$$y_P^{(\alpha)}(x)=\begin{array}{l} \sum_{n=1}^{\infty}\ A_n n^{\alpha}\cos_{\alpha}(n^{\alpha}x^{\alpha}) \\ +\sum_{n=1}^{\infty}\ B_n n^{\alpha}\sin_{\alpha}(n^{\alpha}x^{\alpha}). \end{array} \quad (21)$$

Submitting (20)-(21) into (18), we obtain

$$\begin{array}{l} a(\sum_{n=1}^{\infty}\ A_n n^{\alpha}\cos_{\alpha}(n^{\alpha}x^{\alpha}) \\ +\sum_{k=1}^{\infty}\ B_n n^{\alpha}\sin_{\alpha}(n^{\alpha}x^{\alpha})\big) \\ +b(a_0+\sum_{n=1}^{\infty}\ A_n\sin_{\alpha}(n^{\alpha}x^{\alpha}) \quad (22) \\ +\sum_{n=1}^{\infty}\ B_n\cos_{\alpha}(n^{\alpha}x^{\alpha})\big) \\ =\sum_{n=1}^{\infty}\ \sin_{\alpha}(n^{\alpha}x^{\alpha}). \end{array}$$

Hence, we get

$$\begin{array}{l} a_0 b=0, \\ aA_n n^{\alpha}+bB_n=0, \quad (23) \\ aB_n n^{\alpha}+bA_n=1. \end{array}$$

Therefore, we can calculate

$$\begin{array}{l} a_0=0, \\ A_n=-\frac{b}{a^2n^{2\alpha}-b^2}, \quad (24) \\ B_n=\frac{an^{\alpha}}{a^2n^{2\alpha}-b^2}. \end{array}$$

In view of (24), we give the solution of (18) as follows:

$$y_P(x)=\begin{array}{l} -\sum_{n=1}^{\infty}\ \frac{b}{a^2n^{2\alpha}-b^2}\sin_{\alpha}(n^{\alpha}x^{\alpha}) \\ +\sum_{n=1}^{\infty}\ \frac{an^{\alpha}}{a^2n^{2\alpha}-b^2}\cos_{\alpha}(n^{\alpha}x^{\alpha}). \end{array} \quad (25)$$

Example 2. We now consider the following differential equation on Cantor sets:

$\frac{d^{2\alpha}x}{d^{2\alpha}t} + px = f(t),\ +\infty > t > -\infty,$ (26)

subject to the initial value condition

$\left.\frac{d^{\alpha}x}{d^{\alpha}t}\right|_{t=0} = 0,\ x(0) = 0,$ (27)

where p is constant and $f(t)$ is the local fractional continuous function so that its local fractional Fourier transform exists.
Application of local fractional Fourier transform gives

$-\omega^{2\alpha}x_{\omega}^{F,\alpha}(\omega) + px_{\omega}^{F,\alpha}(\omega) = f_{\omega}^{F,\alpha}(\omega),$ (28)

so that

$(-\omega^{2\alpha} + p)x_{\omega}^{F,\alpha}(\omega) = f_{\omega}^{F,\alpha}(\omega).$ (29)

From (29), we have

$x_{\omega}^{F,\alpha}(\omega) = \frac{f_{\omega}^{F,\alpha}(\omega)}{(-\omega^{2\alpha}+p)}.$ (30)

Therefore, taking the inverse formula of local fractional Fourier transform, we have

$x(t) = -\frac{p^{-(1/2)}}{\Gamma(1+\alpha)}\int_{-\infty}^{t} f(t-\tau)\sin_{\alpha}\left(p^{1/2}\tau^{\alpha}\right)(d\tau)^{\alpha}.$ (31)

Example 3. Let us find the solution to the differential equation on Cantor sets

$\frac{d^{2\alpha}x}{d^{2\alpha}t} + \frac{d^{\alpha}x}{d^{\alpha}t} - 2x = f(t),\ t > 0,$ (32)

subject to the initial value condition

$\left.\frac{d^{\alpha}x}{d^{\alpha}t}\right|_{t=0} = 0,\ x(0) = 0,$ (33)

where $f(t)$ is the local fractional continuous function so that its local fractional Laplace transform exists.

Taking the local fractional Laplace transform, from (32), we have

$$\begin{aligned}&\left(s^{2\alpha}x_{s}^{L,\alpha}(s) - s^{\alpha}x(0) - x^{(\alpha)}(0)\right) + \left(s^{\alpha}x_{s}^{L,\alpha}(s) - x(0)\right)\\ &+2x_{s}^{L,\alpha}(s) = f_{s}^{L,\alpha}(s),\end{aligned} \quad (34)$$

so that

$x_{s}^{L,\alpha}(s) = \frac{f_{s}^{L,\alpha}(s)}{s^{2\alpha}+s^{\alpha}-2}.$ (35)

When the local fractional convolution of two functions is given by [24]

$f_1(t) * f_2(t) = \frac{1}{\Gamma(1+\alpha)}\int_{0}^{t} f_1(t-\tau)f_2(\tau)(d\tau)^{\alpha}$ (36)

and the local fractional Laplace transform of $f_1(t) * f_2(t)$ is [24]

$L_\alpha\{f_1(t) * f_2(t)\} = f_{s,1}^{L,\alpha}(s) f_{s,2}^{L,\alpha}(s)$, (37)

the inverse formula of the local fractional Laplace transform together with the local fractional convolution theorem gives the solution

$x(t) = \frac{1}{\Gamma(1+\alpha)} \int_0^t f(t-\tau)\big(E_\alpha(-2\tau^\alpha) + E_\alpha(\tau^\alpha)\big)(d\tau)^\alpha$. (38)

4. Conclusions

In this study, we explored the relationships between special functions defined on Cantor sets and specific integral transformations employing local fractional calculus techniques. Specifically, we examined the local fractional Fourier series, Fourier transforms, and Laplace transforms. These transformative methods were effectively employed to address three distinct local fractional differential equations, resulting in the identification of solutions characterized by their nondifferentiable properties.

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