A Study of Fractional Calculus Operators Associated with Generalized Functions

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Abstract:

Fractional calculus has emerged as a powerful tool in various scientific and engineering disciplines, offering a deeper understanding of complex phenomena through the analysis of non-integer order derivatives and integrals. In this research paper, we explore the applications of fractional calculus operators in the analysis of special functions and their properties. We begin by providing an overview of fractional calculus and its significance in modern research. Subsequently, we delve into the fundamental concepts of special functions and their representation using fractional derivatives and integrals. The paper presents a comprehensive analysis of well-known special functions, such as the Gamma function, Bessel function, and Hypergeometric functions, and examines how fractional calculus influences their behavior, properties, and applications. Furthermore, we investigate the utilization of fractional calculus in solving differential equations involving special functions, highlighting its implications in various fields, including physics, signal processing, and mathematical modeling.

Introduction

Theory of special functions has a long and varied history with immense literature due to their applications in solving various problems arising in physical, biological and engineering sciences. Special functions have an origin in the solution of partial differential equations satisfying certain prescribed conditions. At present special functions are defined in several ways notably by power series, generating functions, infinite products, integrals, difference equations, trigonometric or orthogonal function series. Eminent mathematicians notably Euler, Legendre, Gauss, Jacobi, Weierstrass, Kummer, Riemann, Ramanujan worked hard to develop special functions like Bessel functions, Whittaker functions, Gauss hypergeometric function and the polynomials that go by the names of Jacobi, Legendre, Laguerre, Hermite etc. The Gaussian hyper geometric function pFq and its special cases are commonly used in applied mathematics and mathematical physics. Since pFq diverges for $p > q + 1$, in an attempt to give meaning to it in this case , MacRobrtand Meijer introduced the E-function and the G-function[2], respectively. Though these functions are quite general in character, a number of special functions like Wright's generalized hypergeometric function[45], generalized parabolic cylinder function, Mittag-Leffler's function, and several other functions, do not form their special cases. Fractional Calculus is a field of mathematic study that grow out of the traditional definitions of the calculus integral and derivatives operators in much the same way fractional exponents is an outgrowth of exponents with integer value. Consider the physical meaning of the exponent. According to our primary school teachers exponents provide a short notation for what is essentially a repeated multiplication of a numerical value. This concept in itself is easy to grasp and straight forward. However, this physical definition can clearly become confused when exponents of non integer value. The fractional derivatives (and fractional integral) of special function of one and more variable is important such as in the evolution of series and integrals the derivation of generating function and the solution of the integral equation motivated by these and many other avenues of applications the fractional calculus operator αDx μ is much used in the theory of special function of one and more variables.The fractional derivative, extension of the ordinary derivative to an non integral value of the order, is of immense utility in finding the solution of ordinary, partial and integral equations, as well as in other contexts. Fractional derivatives for conventional functions were introduced long before the systematic development of the generalized function. Indeed one of the more creditable aspects of the new generalized functional theory would seem to be the fractional derivatives as well as hypergeometric function. A Fractional calculus has its origin in the question of the extension of meaning. A well known example is the extension of meaning of real numbers to complex numbers, and another is the extension of meaning of factorials of integers to factorials of complex numbers. In generalized integration and differentiation the question of the extension of meaning is: Can the meaning of derivatives of integral order d n y be extended to have meaning where n is any number irrational, dxn fractional or complex.Leibnitz invented the above notation. It was naive play with symbols that prompted L'Hospital to ask Leibnitz about the possibility that n be a fraction. "What if n be ½? ", asked L'Hospital. Leibnitz in 1695, replied, "It will lead to a paradox." But he added prophetically, "From this apparent paradox, one day useful consequences will be drawn." In 1697, Leibnitz, referring to Wallis's infinite product for π/2, used the notation d½ y and stated that differential calculus might have been used to achieve the same result. In the past century, many authors have generalized H-function. In a recent paper, Saxena et al. [16] have introduced a generalization of Saxena's I-function [22]. This is also a generalization of Fox's H-function. Saxena and Pogany [17] have studied fractional integral formulae for the Aleph function. Sudland et al. [42] studied the generalized fractional driftless Fokker-Planck equation with power law coefficient. As a result a special function was found, which a particular case of the Aleph function is. The results obtained by the authors serve as the key formulas for numerous potentially useful special functions of Science, Engineering and Technology scattered in the literature. The integral representations involving the product of the Aleph function with exponential function, Gauss hypergeometric function and H-function has been obtained by Sharma [24] and some special cases of the established results are also discussed. The results obtained are useful where the Ifunction occurs naturally.

Objectives

We propose to study in the following areas:

1. To solve fractional kinetic and diffusion equations involving various fractional integral and differential operators using integral transforms.
2. The study of q- analogues of various integral transforms such as q- Laplace transform, qMellin transform, q- Fourier transform, q- Henkel transform etc. and find out their possible new applications.

3. To study of q-analogues on various generalization of fractional differential and integral operators

Methodology

In the present study, it is proposed:

1. To investigate the fractional calculus operators of the generalized Mittag-Leffler type functions
2. To derive the recurrence relations and integral representations of the generalized Mittag-Leffler type functions.
3. To generalize the functions of fractional calculus and derive the relations that exists among these functions and fractional calculus operators.
4. To use the integral transforms for finding the solutions of the fractional differential and integral equations.
5. To find the solution of fractional differential and integral equations arising in physical, biological and engineering sciences.
6. To deduce the results for the basic hypergeometric functions as an application of fractional q Calculus

Some of the important fractional calculus operators are as follows:

Riemann- Liouville left-sided fractional integral of order α

$$ {}_aI_x^{\alpha} f(x) = {}_aD_x^{-\alpha} f(s) = I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)\,dt}{(x-t)^{1-\alpha}}, \quad x > a \tag{1} $$

Riemann- Liouville right-sided fractional integral of order α

$$ {}_xI_b^{\alpha} f(x) = {}_xD_b^{-\alpha} f(s) = I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)\,dt}{(x-t)^{1-\alpha}}, \quad x < b \tag{2} $$

Riemann- Liouville left-sided fractional derivative of order α

$$ {}_aD_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x \frac{f(t)\,dt}{(x-t)^{\alpha-n+1}}, \quad (n = [\alpha]+1) \tag{3} $$

Where $[\alpha]$ denotes the integral part of α .

Riemann- Liouville right-sided fractional derivative of order α

$$ {}_xD_b^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n \int_x^b \frac{f(t)\,dt}{(x-t)^{\alpha-n+1}}, \quad (n = [\alpha]+1) \tag{4} $$

Where $[\alpha]$ denotes the integral part of α .

Caputo fractional derivative

$$ {}_0^{c}D_x^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{f(t)\,dt}{(x-t)^{\alpha}}, \tag{5} $$

Where $0 < \alpha < 1$.

Some integral transforms which will be used in the research work:

Laplace transform

The Laplace transform of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $F(s)$, defined by:

$$ F(s) = L\{f(t)\} = \int_0^{\infty} e^{-st} F(t)\,dt \tag{6} $$

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