Inequalities of the Hermite-Hadmard Type for the Product of Functions Through The Application of Convex Functions

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Abstract

One of the several approaches for generating a novel convex function from given functions involves multiplying these functions while imposing specific conditions on them. Typically, the product of two or a finite number of convex functions may not inherently be convex, prompting a deeper investigation into the nature of these products. In this research, we redefine the concept of function multiplication within the framework of generalized convex functions to establish inequalities of the Hermite-Hadamard type for these functions. We have conducted a thorough analysis, considering various scenarios involving double and triple integrals, leading to the derivation of fresh findings. These presented results can be considered as enhancements and refinements of previously established findings.

Introduction

The theory of convex functions plays a fundamental role across various branches of mathematics, notably in optimization and contemporary analysis. Convex functions exhibit numerous distinctive characteristics; for instance, in the case of strict convexity, such functions possess a unique minimum within an open set. Remarkably, even in spaces of infinite dimension, convex functions retain analogous properties, rendering them as exemplary functionals in variation methods. In probability theory, convex functions associated with random variables are bounded above by their expected values, a principle known as Jensen's inequality. This inequality can be employed to rediscover various other inequalities, including the arithmetic-geometric mean inequality and Hölder's inequality. Among the well-established results in the realm of convexity, the renowned Hermite-Hadamard inequality stands out as a pivotal example.

$$p\left(\frac{\xi_1+\xi_2}{2}\right) \leq \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2} p(x)\mathrm{d}x \leq \frac{p(\xi_1)+p(\xi_2)}{2}. \quad (1)$$

Convexity has seen broad generalization across various aspects, with the classical Hermite-Hadamard inequality serving as a reference point for these generalizations. Toader expanded the notion of convexity by introducing the concept of an m-convex function, along with several results, including Hermite-Hadamard-type inequalities. Subsequent progress in the field has been observed regarding m-convex and (α,m)-convex functions, as evident from references in the literature. Furthermore, advancements have been made in understanding convex, m-convex, and (α,m,h)-convex functions.

Pachpatte explored the product of functions as a means to develop Hermite-Hadamard-type inequalities, relying on traditional convexity principles. More recently, Noor et al. introduced the concept of (α,m,h)-convex functions and established fundamental inequalities for twice-differentiable functions falling within this category. Since (α,m,h)-convexity encompasses classical convexity, m-convexity, and (α,m)-convex functions, their results can be considered as extensions of these earlier findings. Moreover, these investigations have highlighted the elegant relationship between convex functions and their various generalizations through the use of function products.

Motivated by these generalizations, we apply the concept of function products within the framework of (α,m,h)-convex functions to derive Hermite-Hadamard-type inequalities for functions. Our research explores diverse scenarios involving double and triple integrals, resulting in the discovery of novel outcomes. This innovative approach allows us to characterize convex functions by leveraging products of h-convex, m-convex, (s,m)-convex, and (α,s)-convex functions. Consequently, our findings provide a unified perspective on several classes of functions, such as (s,m)-Godunova-Levin, m-Godunova-Levin, and (α,s)-Godunova-Levin functions, all with relatively mild conditions.

2. Preliminaries

This section is devoted to few well-known definitions fromthe literature. In [1], Toader gave the definition of m-convex function in the following manner.

Definition 1. A function $p: I \longrightarrow (0, \infty)$ is m-convex, if

$p(\eta x_1 + m(1-\eta)x_2) \leq tp(x_1) + m(1-\eta)p(x_2)$. (2)

for all $x_1, x_2 \in I, m \in [0,1]$, and $\eta \in [0,1]$.

Mihesan extended the idea of m-convex functions by introducing the idea of (α, m)-convex function as follows.

Definition 2. A function $p: I \longrightarrow (0, \infty)$ is (α, m)-convex, if

$p(\eta x_1 + m(1-\eta)x_2) \leq \eta^\alpha p(x_1) + m(1-\eta^\alpha)p(x_2)$, (3)

for all $x_1, x_2 \in I, \eta \in [0,1]$, and $(\alpha, m) \in [0,1]^2$.

Noor et al. generalized the idea of m-convexity and (α, m)-convexity in a more general way with the definition of (α, m, h)-convexity.

Definition 3. A function $p: I \longrightarrow (0, \infty)$ is (α, m, h)-convex, if

$p(\eta x_1 + m(1-\eta)x_2) \leq h(\eta^\alpha)p(x_1) + mh(1-\eta^\alpha)p(x_2)$, (4)

for all $x_1, x_2 \in I, \eta \in [0,1], (\alpha, m) \in [0,1]^2$ and $h: [0,1] \longrightarrow [0,1]$.

Pachpatte used the idea of product of functions for convex functions to establish the following result.

Theorem 1. Let p and q be nonnegative and convex functions on $[\xi_1, \xi_2]$ and further assume that they are real valued. Then, the following two inequalities hold:

$$\begin{aligned} \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2} p(x)q(x)\mathrm{d}x \leq & \ \frac{1}{3}M(\xi_1,\xi_2)+\frac{1}{6}N(\xi_1,\xi_2), \\ 2p\left(\frac{\xi_1+\xi_2}{2}\right)q\left(\frac{\xi_1+\xi_2}{2}\right) \leq & \ \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2} p(x)q(x)\mathrm{d}x \qquad (5)\ (6) \\ & +\frac{1}{6}M(\xi_1,\xi_2)+\frac{1}{3}N(\xi_1,\xi_2). \end{aligned}$$

3. Main Results

This section contains the main results of our work involving product of (α, m, h)-convex functions to obtain Hermite-Hadamard-type integral inequalities for two functions p and q. These inequalities have also been studied for double and triple integrals.

Theorem 2. Assume that p and q are nonnegative and realvalued functions with $pq \in L[\xi_1, \xi_2]$, where $\xi_1, b \in I$ and $\xi_1 < \xi_2$. Furthermore, assume that p and q are (α, m, h)-convex on $[\xi_1, \xi_2]$. Then, we have the following inequality:

$$\frac{1}{m\xi_2-\xi_1}\int_{\xi_1}^{m\xi_2} p(x)q(x)\mathrm{d}x \leq h(\xi_1,\xi_1)L + h(\xi_2,\xi_2)M + h(\xi_1,\xi_2)N. \ (7)$$

Here,

$$\begin{aligned} H(\xi_1,\xi_1) &= p(\xi_1)q(\xi_1), \\ H(\xi_2,\xi_2) &= p(\xi_2)q(\xi_2), \qquad (8) \\ H(\xi_1,\xi_2) &= p(\xi_1)q(\xi_2)+p(\xi_2)q(\xi_1). \end{aligned}$$

Also, $L = \int_0^1 [h(\eta^\alpha)]^2 d\eta, M = \int_0^1 m^2[h(1-\eta^\alpha)]^2 d\eta$ and $N = \int_0^1 mh(\eta^\alpha)h(1-\eta^\alpha)d\eta$.

Proof. Assume that p and q are (α, m, h)-convex on $[\xi_1, \xi_2]$; then,

$$\begin{aligned} p(\eta\xi_1+m(1-\eta)\xi_2)q(\eta\xi_1+m(1-\eta)\xi_2) \leq & \ [h(\eta^\alpha)p(\xi_1)+mh(1-\eta^\alpha)p(\xi_2)][h(\eta^\alpha)q(\xi_1)+mh(1-\eta^\alpha)q(\xi_2)] \\ = & \ [h(\eta^\alpha)]^2p(\xi_1)q(\xi_1)+m^2[h(1-\eta^\alpha)]^2p(\xi_2)q(\xi_2)+mh(\eta^\alpha)\times \\ & \ h(1-\eta^\alpha)[p(\xi_1)q(\xi_2)+p(\xi_2)q(\xi_1)]. \end{aligned}$$

(9)

Now,

$$\begin{aligned} & \int_0^1 p(\eta\xi_1+m(1-\eta)\xi_2)q(\eta\xi_1+m(1-\eta)\xi_2)\mathrm{d}\eta \\ & \leq \left[\begin{array}{c} \int_0^1 [h(\eta^\alpha)]^2p(\xi_1)q(\xi_1)d\eta + \int_0^1 m^2[h(1-\eta^\alpha)]^2p(\xi_2)q(\xi_2)\mathrm{d}\eta \\ +\int_0^1 mh(\eta^{\mathrm{a}})h(1-\eta^{\mathrm{a}})[p(\xi_1)q(\xi_2)+p(\xi_2)q(\xi_1)]\mathrm{d}\eta \end{array}\right]. \ (10) \end{aligned}$$

Thus,

$$\int_0^1 p(\eta\xi_1 + m(1-\eta)\xi_2)q(\eta\xi_1 + m(1-\eta)\xi_2)\mathrm{d}\eta \leq \begin{bmatrix} h(\xi_1,\xi_1)L + h(\xi_2,\xi_2)M \\ +h(\xi_1,\xi_2)N \end{bmatrix}. \quad (11)$$

$$\frac{1}{m\xi_2-\xi_1}\int_{\xi_1}^{m\xi_2} p(x)q(x)\mathrm{d}x \leq \begin{matrix} H(\xi_1,\xi_1)L + H(\xi_2,\xi_2)M \\ +H(\xi_1,\xi_2)N. \end{matrix} \quad (12)$$

Now, substituting $\eta\xi_1 + m(1-\eta)\xi_2 = x$, we obtain

Corollary 1. If $h(\eta) = \eta$, then

$$\frac{1}{m\xi_2-\xi_1}\int_{\xi_1}^{m\xi_2} p(x)q(x)\mathrm{d}x \leq \begin{bmatrix} \frac{1}{2\alpha+1}H(\xi_1,\xi_1) + \frac{2m^2\alpha^2}{(2\alpha+1)(\alpha+1)}H(\xi_2,\xi_2) \\ +\frac{m\alpha}{(2\alpha+1)(\alpha+1)}H(\xi_1,\xi_2) \end{bmatrix}. \quad (13)$$

Remark 1. Corollary 1 , along with $m = 1$ and $\alpha = 1$, gives inequality (5).

Theorem 3. For any two (α, m, h)-convex functions p and q with $pq \in L[\xi_1, \xi_2]$, we have the following estimates:

$$2p\left(\frac{\xi_1+m\xi_2}{2}\right)q\left(\frac{\xi_1+m\xi_2}{2}\right) \leq \begin{bmatrix} \int_0^1 p(\eta\xi_1 + n(1-\eta)\xi_2)q(\eta\xi_1 + m(1-\eta)\xi_2)\mathrm{d}\eta \\ +[H(\xi_1,\xi_1) + n^2H(\xi_2,\xi_2)]S + \frac{1}{2}[H(\xi_1,\xi_2)]T \end{bmatrix}, \quad (14)$$

where

$$\begin{aligned} S &= \int_0^1 h(\eta^\alpha)h(1-\eta^\alpha)\mathrm{d}\eta, \\ T &= \int_0^1 n[[h(\eta^\alpha)]^2 + [h(1-\eta^\alpha)]^2]\mathrm{d}\eta. \end{aligned} \quad (15)$$

Also $H(\xi_1,\xi_1), H(\xi_2,\xi_2)$, and $H(\xi_1,\xi_2)$ are as defined in the above theorem. Proof. As p and q are (α, m, h)-convex, we have

$$\begin{aligned} p\left(\frac{\xi_1+m\xi_2}{2}\right)q\left(\frac{\xi_1+m\xi_2}{2}\right) &= \begin{bmatrix} p\left(\frac{\eta\xi_1+m(1-\eta)\xi_2}{2} + \frac{(1-\eta)\xi_1+m\eta\xi_2}{2}\right)\times \\ q\left(\frac{\eta\xi_1+m(1-\eta)\xi_2}{2} + \frac{(1-\eta)\xi_1+m\eta\xi_2}{2}\right) \end{bmatrix} \\ &\leq \begin{bmatrix} \frac{1}{4}[p(\eta\xi_1 + m(1-\eta)\xi_2) + p((1-\eta)\xi_1 + m\eta\xi_2)]\times \\ [q(\eta\xi_1 + m(1-\eta)\xi_2 + q((1-\eta)\xi_1 + m\eta\xi_2))] \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} \frac{1}{4}[p(\eta\xi_1 + m(1-\eta)\xi_2)q(\eta\xi_1 + m(1-\eta)\xi_2) \\ +p((1-\eta)\xi_1 + m\eta\xi_2)q((1-\eta)\xi_1 + m\eta\xi_2)] \\ +\frac{1}{4}[p(\eta\xi_1 + m(1-\eta)\xi_2)q((1-\eta)\xi_1 + m\eta\xi_2) \\ +p((1-\eta)\xi_1 + m\eta\xi_2)q(\eta\xi_1 + m(1-\eta)\xi_2)] \end{bmatrix}$$

$$\leq \begin{bmatrix} \frac{1}{4}[p(\eta\xi_1 + m(1-\eta)\xi_2)q(\eta\xi_1 + m(1-\eta)\xi_2)] \\ +p((1-\eta)\xi_1 + m\eta\xi_2)q((1-\eta)\xi_1 + m\eta\xi_2)] \\ +\frac{1}{4}[h(\eta^\alpha)p(\xi_1) + mh(1-\eta^\alpha)p(\xi_2)] \times \\ [h(1-\eta^\alpha)q(\xi_1) + mh(\eta^\alpha)q(\xi_2)] \end{bmatrix} \quad (16)$$

$$\leq \begin{bmatrix} \frac{1}{4}[p(\eta\xi_1 + m(1-\eta)\xi_2)q(\eta\xi_1 + m(1-\eta)\xi_2) + p((1-\eta)\xi_1 \\ +m\eta\xi_2)q((1-\eta)\xi_1 + m\eta\xi_2)] \\ +\frac{1}{2}[h(\eta^\alpha)h(1-\eta^\alpha)][p(\xi_1)p(\xi_2) + m^2p(\xi_2)q(\xi_2)] \\ +\frac{1}{4}m[h(\eta^\alpha)]^2 + [h(1-\eta^\alpha)]^2\Big][p(\xi_1)q(\xi_2) + p(\xi_2)q(\xi_1)] \end{bmatrix}$$

On integrating

$$p\left(\frac{\xi_1+m\xi_2}{2}\right)q\left(\frac{\xi_1+m\xi_2}{2}\right) \leq \begin{bmatrix} \frac{1}{4}\int_0^1 [p(\eta\xi_1 + m(1-\eta)\xi_2)q(\eta\xi_1 + m(1-\eta)\xi_2) \\ +p((1-\eta)\xi_1 + m\eta\xi_2)q((1-\eta)\xi_1 + m\eta\xi_2)]\mathrm{d}\eta \\ +\frac{1}{2}\int_0^1 h(\eta^\alpha)h(1-\eta^\alpha)\mathrm{d}\eta[p(\xi_1)p(\xi_2) + m^2p(\xi_2)q(\xi_2)] \\ +\frac{1}{4}\int_0^1 [m[h(\eta^\alpha)]^2] + [h(1-\eta^\alpha)]^2\,\mathrm{d}\eta[p(\xi_1)q(\xi_2) + p(\xi_2)q(\xi_1)] \end{bmatrix} (17)$$

Thus, we obtain

$$p\left(\frac{\xi_1+m\xi_2}{2}\right)q\left(\frac{\xi_1+m\xi_2}{2}\right) \leq \begin{bmatrix} \frac{1}{2}\int_0^1 p(\eta\xi_1 + m(1-\eta)\xi_2)q(\eta\xi_1 + m(1-\eta)\xi_2)\mathrm{d}\eta \\ 1\frac{1}{2}[H(\xi_1,\xi_1) + mH(\xi_2,\xi_2)]S + \frac{1}{4}[H(\xi_1,\xi_2)]T \end{bmatrix}. (18)$$

Now, substituting $x = \eta\xi_1 + n(1-\eta)\xi_2$, we obtain

$$2p\left(\frac{\xi_1+m\xi_2}{2}\right)q\left(\frac{\xi_1+m\xi_2}{2}\right) \leq \left[\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_1} p(x)q(x)\mathrm{d}\eta + [h(\xi_1,\xi_1) + nh(\xi_2,\xi_2)]S + \frac{1}{2}[h(\xi_1,\xi_2)]T\right]. (19)$$

Corollary 2. If $h(\eta) = \eta$, then

$$2p\left(\frac{\xi_1+m\xi_2}{2}\right)q\left(\frac{\xi_1+m\xi_2}{2}\right) \leq \begin{bmatrix} \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2} p(x)q(x)\mathrm{d}\eta + \frac{\alpha}{(\alpha+1)(2\alpha+1)}[H(\xi_1,\xi_1) + mH(\xi_2,\xi_2)] + \frac{m(\alpha^2+\alpha+1)}{2(\alpha+1)(2\alpha+1)} \\ +\frac{m(\alpha^2+\alpha+1)}{2(\alpha+1)(2\alpha+1)}H(\xi_1,\xi_2) \end{bmatrix}.$$

(20)

Remark 2. Corollary 2, along with $m=1$ and $\alpha=1$, gives inequality (6).
Theorem 4. Assume that p and q are (α, m, h)-convex functions satisfying all the conditions of the above theorem; then, the following inequality holds:

$$\begin{aligned}&\frac{1}{(\xi_2-\xi_1)^2}\int_{\xi_1}^{\xi_2}\int_{\xi_1}^{\xi_2}\int_0^1 p(\eta x+m(1-\eta)y)q(\eta x+m(1-\eta)y)\mathrm{d}\eta\mathrm{d}y\,\mathrm{d}x\\&\leq[L+M]\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2} p(x)q(x)\mathrm{d}x+\frac{1}{2}\frac{1}{(\xi_2-\xi_1)^2}N[H(\xi_1,\xi_1)+H(\xi_2,\xi_2)+H(\xi_1,\xi_2)],\end{aligned} \quad (21)$$

where

$$\begin{aligned}L&=\int_0^1 [h(\eta^\alpha)]^2\,\mathrm{d}\eta,\\M&=\int_0^1 m^2[h(1-\eta^\alpha)]^2\,\mathrm{d}\eta, \quad (22)\\N&=\int_0^1 mh(\eta^\alpha)h(1-\eta^\alpha)\mathrm{d}\eta.\end{aligned}$$

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Integrating on [0,1], we obtain

$$\begin{aligned}\int_0^1 p(\eta x+n(1-\eta)y)q(\eta x+m(1-\eta)y)d\eta \leq\ & p(x)q(x)\int_0^1 [h(\eta^\alpha)]^2\,\mathrm{d}\eta+p(y)q(y)\int_0^1 m^2[h(1-\eta^\alpha)]^2\,\mathrm{d}\eta\\&+[p(x)q(y)+p(y)q(x)]\int_0^1 mh(\eta^a)h(1-\eta^\alpha)\mathrm{d}\eta.\end{aligned}$$

(24)

Thus,
Now, integrating on the rectangle $[0,1]\times[0,1]$,

$$\begin{aligned}&\int_0^1 p(\eta x+m(1-\eta)y)q(\eta x+m(1-\eta)y)\mathrm{d}\eta\\&\leq[p(x)q(x)]L+[p(y)q(y)]M+[p(x)q(y)+p(y)q(x)]N.\end{aligned} \quad (25)$$

$$\begin{aligned}&\int_{\xi_1}^{\xi_1}\int_{\xi_1}^{\xi_2}\int_0^1 p(\eta x+m(1-\eta)y)q(\eta x+m(1-\eta)y)\mathrm{d}\eta\mathrm{d}y\,\mathrm{d}x\\&\leq L(\xi_2-\xi_1)\int_{\xi_1}^{\xi_2} p(x)q(x)\mathrm{d}x+M(\xi_2-\xi_1)\int_{\xi_1}^{\xi_1} p(y)q(y)\mathrm{d}y+N\left[\int_{\xi_1}^{x_2} p(x)\mathrm{d}x\times\int_{\xi_1}^{\xi_2} q(y)\mathrm{d}y\right.\\&\left.+\int_{\xi_1}^{\xi_2} p(y)\mathrm{d}y\times\int_{\xi_1}^{\xi_2} q(x)\mathrm{d}x\right].\end{aligned}$$

(26)

Now, applying Hadamard's inequality from right half to the above equation,

$$\int_{\xi_1}^{\xi_2}\int_{\xi_1}^{\xi_2}\int_0^1 p(\eta x+m(1-\eta)y)q(\eta x+m(1-\eta)y)\mathrm{d}\eta\mathrm{d}y\,\mathrm{d}x$$
$$\leq [L+M](\xi_2-\xi_1)\int_{\xi_1}^{\xi_2} p(x)q(x)\mathrm{d}x+\frac{1}{2}N[p(\xi_1)q(\xi_1)+p(\xi_2)q(\xi_2)+p(\xi_1)q(\xi_2)+p(\xi_2)q(\xi_1)]$$
$$= [L+M](\xi_2-\xi_1)\int_{\xi_1}^{\xi_2} p(x)q(x)\mathrm{d}x+\frac{1}{2}N[H(\xi_1,\xi_1)+H(\xi_2,\xi_2)+H(\xi_1,\xi_2)].$$
(27)

Thus, we obtain

$$\frac{1}{(\xi_2-\xi_1)^2}\int_{\xi_1}^{\xi_2}\int_{\xi_1}^{\xi_2}\int_0^1 p(\eta x+m(1-\eta)y)q(\eta x+m(1-\eta)y)\mathrm{d}\eta\mathrm{d}y\,\mathrm{d}x$$
$$\leq \frac{1}{\xi_2-\xi_1}[L+M]\int_{\xi_1}^{\xi_2} p(x)q(x)\mathrm{d}x+\frac{1}{2}\frac{1}{(\xi_2-\xi_1)^2}N[H(\xi_1,\xi_1)+H(\xi_2,\xi_2)+H(\xi_1,\xi_2)]. \quad (28)$$

Corollary 3. For $h(\eta)=\eta$, we have

$$\frac{1}{(\xi_2-\xi_1)}\int_{\xi_1}^{2}\int_{\xi_1}^{\xi_2}\int_0^{\xi_2} p(\eta x+m(1-\eta y)q(\eta x+n(1-\eta y)\mathrm{d}\eta\mathrm{d}y\,\mathrm{d}x$$
$$\leq \left[\begin{array}{c}\frac{2\alpha^2m^2+\alpha+1}{(2\alpha+1)(\alpha+1)(\xi_2-\xi_1)}\int_{\xi_1}^{\xi_2} p(x)q(x)\mathrm{d}x\\ +\frac{m\alpha}{2(2\alpha+1)(\alpha+1)(\xi_2-\xi_1)^2}[H(\xi_1,\xi_1)+H(\xi_2,\xi_2)+H(\xi_1,\xi_2)]\end{array}\right]. \quad (29)$$

R**emark 3.** For $\alpha=1=m$, we obtain equation (3).

Theorem 5. Assume that p and q are nonnegative realvalued functions such that they are (α,m,h)-convex and $pq\in L[\xi_1,\xi_2 l$; then,

$$\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\int_0^1 p\left(\eta x+m(1-\eta)\frac{\xi_1+\xi_2}{2}\right)q\left(\eta x+m(1-\eta)\frac{\xi_1+\xi_2}{2}\right)\mathrm{d}\eta\mathrm{d}x$$
$$\leq \frac{L}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2} p(x)q(x)\mathrm{d}x+\frac{1}{4}M[H(\xi_1,\xi_1)+H(\xi_2,\xi_2)+H(\xi_1,\xi_2)]+\frac{N}{2(\xi_2-\xi_1)}[H(\xi_1,\xi_1)+H(\xi_2,\xi_2)+H(\xi_1,\xi_2)],$$
(30)

where L,M, and N are as in the above theorem.

Proof. Since p and q are (α,m,h)-convex,

$$p\left(\eta x+m(1-\eta)\left(\frac{\xi_1+\xi_2}{2}\right)\right)q\left(\eta x+m(1-\eta)\left(\frac{\xi_1+\xi_2}{2}\right)\right)$$
$$\leq\left[h(\eta^\alpha)p(x)+mh(1-\eta^\alpha)p\left(\frac{\xi_1+\xi_2}{2}\right)\right]\left[h(\eta^\alpha)q(x)+mh(1-\eta^\alpha)q\left(\frac{\xi_1+\xi_2}{2}\right)\right] \quad (31)$$
$$=\left[[h(\eta^\alpha)]^2p(x)q(x)+m^2[h(1-\eta^\alpha)]^2p\left(\frac{\xi_1+\xi_2}{2}\right)q\left(\frac{\xi_1+\xi_2}{2}\right)\right]$$
$$+mh(\eta^\alpha)h(1-\eta^\alpha)\left[p(x)q\left(\frac{\xi_1+\xi_2}{2}\right)+q(x)p\left(\frac{\xi_1+\xi_2}{2}\right)\right].$$

This implies that

$$\int_0^1 p\left(\eta x+n(1-\eta)\left(\frac{\xi_1+\xi_2}{2}\right)\right) q\left(\eta x+m(1-\eta)\left(\frac{\xi_1+\xi_2}{2}\right)\right) \mathrm{d}\eta$$
$$\leq p(x)q(x)\int_0^1 [h(\eta^\alpha)]^2\,\mathrm{d}\eta + p\left(\frac{\xi_1+\xi_2}{2}\right) q\left(\frac{\xi_1+\xi_2}{2}\right)\int_0^1 m^2[h(1-\eta^\alpha)]^2\,\mathrm{d}\eta \quad (32)$$
$$+\left[p(x)q\left(\frac{\xi_1+\xi_2}{2}\right)+q(x)p\left(\frac{\xi_1+\xi_2}{2}\right)\right]\int_0^1 mh(\eta^\alpha)h(1-\eta^\alpha)\mathrm{d}\eta.$$

Integrating on [0,1],

$$\int_{\xi_1}^{\xi_1}\int_0^1 p\left(\eta x+n(1-\eta)\left(\frac{\xi_1+\xi_2}{2}\right)\right) q\left(\eta x+m(1-\eta)\left(\frac{\xi_1+\xi_2}{2}\right)\right) \mathrm{d}\eta\mathrm{d}x$$
$$\leq L\int_{\xi_1}^{\xi_2} p(x)q(x)\mathrm{d}x + M(\xi_2-\xi_1)p\left(\frac{\xi_1+\xi_2}{2}\right)q\left(\frac{\xi_1+\xi_2}{2}\right)+\left[q\left(\frac{\xi_1+\xi_2}{2}\right)\int_{\xi_1}^{\xi_2} p(x)\mathrm{d}x \quad (33)\right.$$
$$\left.+\,p\left(\frac{\xi_1+\xi_2}{2}\right)\int_{\xi_1}^{\xi_2} q(x)\mathrm{d}x\right]N.$$

Now, using the right half of Hadamard's inequality on the above equation,

$$\int_{\xi_1}^{\xi_1}\int_0^1 p\left(\eta x+n(1-\eta)\left(\frac{\xi_1+\xi_2}{2}\right)\right) q\left(\eta x+m(1-\eta)\left(\frac{\xi_1+\xi_2}{2}\right)\right) \mathrm{d}v\,\mathrm{d}x$$
$$\leq L\int_{\xi_1}^{\xi_1} p(x)q(x)\mathrm{d}x + M(\xi_2-\xi_1)\left[\frac{p(\xi_1)+p(\xi_2)}{2}\frac{q(\xi_1)+q(\xi_2)}{2}\right] \quad (34)$$
$$+\left[2\frac{p(\xi_1)+p(\xi_2)}{2}\frac{q(\xi_1)+q(\xi_2)}{2}\right]N.$$

Hence, we obtain

$$\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\int_0^1 p\left(\eta x+m(1-\eta)\left(\frac{\xi_1+\xi_2}{2}\right)\right) q\left(\eta x+n(1-\eta)\left(\frac{\xi_1+\xi_2}{2}\right)\right) \mathrm{d}\eta\mathrm{d}x$$
$$\leq \frac{L}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2} p(x)q(x)\mathrm{d}x + \frac{M}{4}[H(\xi_1,\xi_1)+H(\xi_2,\xi_2)+H(\xi_1,\xi_2)] \quad (35)$$
$$+\frac{N}{2(\xi_2-\xi_1)}[h(\xi_1,\xi_1)+h(\xi_2,\xi_2)+h(\xi_1,\xi_2)].$$

This completes the proof.

Corollary 4. If $h(\eta)=\eta$, then

$$\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\int_0^1 p\left(\eta x+n(1-\eta)\left(\frac{\xi_1+\xi_2}{2}\right)\right) q\left(\eta x+n(1-\eta)\left(\frac{\xi_1+\xi_2}{2}\right)\right) \mathrm{d}\eta\mathrm{d}x$$
$$\leq \frac{1}{(\xi_2-\xi_1)(2\alpha+1)}\int_{\xi_1}^{\xi_2} p(x)q(x)\mathrm{d}x + \frac{\alpha^2m^2}{2(\alpha+1)(2\alpha+1)}[h(\xi_1,\xi_1)+h(\xi_2,\xi_2)+h(\xi_1,\xi_2)] \quad (36)$$
$$+\frac{m\alpha}{2(\xi_2-\xi_1)(\alpha+1)(2\alpha+1)}[h(\xi_1,\xi_1)+h(\xi_2,\xi_2)+h(\xi_1,\xi_2)].$$

Remark 4. For $\alpha=1=m$, we obtain equation (4)

4. Conclusion

In this research paper, we leverage the concept of function products to establish a class of generalized convex functions derived from two given functions. Specifically, we explore this product within the context of (α,m,h)-convex functions and subsequently employ it to investigate various types of Hermite-Hadamard inequalities. Notably, we find that when the value of "h" corresponds to the identity function, these results align with those obtained for the product of (α,m)-convex functions. Furthermore, these results hold true when considering the product in the context of m-convexity with α equal to 1. This comparison underscores that the outcomes we obtain represent an enhancement and generalization of the results pertaining to convex, m-convex, and (α,m)-convex functions in a distinctive manner.

For readers interested in further exploration, one may extend this product concept to invex, preinvex, m-preinvex, harmonically preinvex, and logarithmically preinvex functions. Additionally, this idea holds potential significance when applied to fractional integrals and stochastic processes involving convex functions, offering fresh perspectives in this domain.

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