Deriving RL-Monoids Through Subtraction

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Abstract

This paper aims to introduce the concept of subtractive derivations and explore their algebraic characteristics within the context of RL-monoids. Additionally, we provide certain descriptions of subtractive derivations within the Godel center. Furthermore, we identify Godel algebras through a fixed set of subtractive derivations. Lastly, we examine the interplay between subtractive derivations and other types of derivations in RL-monoids. These findings contribute to our understanding of the shared properties of subtractive derivations in t-norm-based fuzzy logical algebras.

Introduction

Residuated lattice ordered monoids (abbreviated as "RL-monoids") were first introduced by Swamy as a unifying concept encompassing both Abelian lattice ordered groups and Heyting algebras. Additionally, RL-monoids are closely connected to algebras in t-norm-based fuzzy logics, with BL algebras and MV-algebras being specific instances of such algebras. It's important to highlight that many properties characteristic of BL-algebras apply to all RL-monoids. Consequently, RL-monoids can be considered as an algebraic framework for a more comprehensive logic than Hájek's basic fuzzy logic, underscoring their significance in the study of fuzzy logic.

The concept of derivations plays a pivotal role in exploring the properties and structures of fuzzy logical algebraic systems. Posner, in 1957, investigated various types of derivations in prime rings along with their fundamental algebraic characteristics. Subsequently, Borzooei et al. provided characterizations of p-semisimple BCI-algebras through derivations with respect to BCI-algebras featuring derivations. In 2008, Xin et al. characterized modular and distributive lattices using isotone derivations in lattices with derivations. Furthermore, Alshehri et al. delved into derivations on MV-algebras, outlining conditions under which an additive derivation is also isotone for a linearly ordered MV-algebra. In 2013, Lee et al. introduced and studied derivations and f-derivations in lattice implication algebras, exploring their relationships with filters. In 2016, He et al. investigated different types of derivations in residuated lattices and provided characterizations of Heyting algebras in terms of these derivations. In 2017, Hua studied derivations in R_0-algebras, which are equivalent to NM-algebras, and examined the connection between filters and the fixed point set of these derivations. Lastly, in 2022, Liu conducted a study on implicative derivations in MTL-algebras and provided characterizations of them based on these types of derivations.

This paper is motivated by the following considerations: prior research on derivations in t-norm-based fuzzy logical algebras has mainly focused on multiplicative derivations and implicative derivations, which are two specific types of maps that satisfy certain properties.

$d(x \otimes y) = (d(x) \oplus y) \uplus (x \circledast d(y))$, (multiplicative derivation),
$d(x \otimes y) = (d(x) \hookrightarrow y)((x \hookrightarrow d(y))$, (implicative derivation). (1)

However, there has been limited research conducted on derivations defined in conjunction with the "m" operation and various other operations within the realm of residuated structures. However, this aspect warrants investigation, as it offers the potential for a more comprehensive exploration within algebraic structures through the incorporation of additional operations. Consequently, it becomes intriguing to delve into the examination of these types of derivations within the context of fuzzy logical algebras.

Taking these considerations into account, we propose a novel form of derivation known as "subtractive derivation" for RL-monoids and conduct an exploration of certain algebraic properties associated with them. The structure of this paper unfolds as follows: In Section 2, we provide an overview of fundamental concepts and definitions pertinent to RL-monoids. Section 3 introduces the concept of subtractive derivation within RL-monoids and offers several characterizations of these derivations. In Section 4, we delve into the relationship between the fixed point set of subtractive derivations and the ideals within RL-monoids. Finally, in Section 5, we examine the connections between subtractive derivations and other types of derivations, such as multiplicative derivations and implicative derivations, within the context of RL-monoids.

2.Preliminaries

First, some basic notions of Rℓ-monoids and their related algebraic results are presented.

Definition 1 (see [9]). An algebra $(\mathcal{H}, \cdot, \hookrightarrow, \mathbb{G}, \mathbb{W}, 0, 1)$ is said to be a residuated lattice if (1) $(\mathcal{M}, \mathbf{m}, w, 0, 1)$ is a bounded lattice,

(2) $(\mathcal{A}, \oplus, 1)$ is a commutative monoid,

(3) $u \oplus v \leq w$ iff $u \leq v \hookrightarrow w$, for any $u, v, w \in L$.

By $\mathcal{A}$ we mean that the universe of a residuated lattice $(\mathcal{M}, \oplus, \hookrightarrow, \mathbb{I}, w, 0, 1)$. On $\mathcal{M}$, we define

$u \leq v$ iff $u \hookrightarrow v = 1$. (2)

Then, $\leq$ is a binary partial order on $\mathcal{M}$ and for $u \in \mathcal{M}$, $0 \leq \bar{u} \leq 1$.

A residuated lattice $\mathcal{H}$ is an $R\ell$-monoid if it satisfies the divisibility equation (DIV) u 角 $v = u \Leftrightarrow (u \hookrightarrow v)$. (3)

An $R\ell$-monoid $\mathcal{U}$ is a Godel algebra if it satisfies

(IDE) $u \oplus u = u$. (4)

We denote the set $\{u \mid u \oplus u = u\}$ of $\mathcal{M}$ by $\mathcal{F}(\mathcal{M})$. In every $R\ell$-monoid, we define the operation as follows:

$u \boxminus v = u \oplus v^{\star}$, (5)

where $v^{\star} = v \hookrightarrow 0$.

Proposition 1 (see [1]). The following hold in $R\ell$-monoid $\mathcal{M}$, for all $u, v, w \in \mathcal{M}$:

(1)$u \boxminus 0 = u, 0 \boxminus u = 0, u \boxminus u = 0, 1 \boxminus u = u^{*}, u \boxminus 1 = 0$,
(2)$u \hookrightarrow (v \to w) = (u \oplus v) \hookrightarrow w = v \hookrightarrow (u \hookrightarrow w)$,
(3) if $u \leq v$, then $v \hookrightarrow w \leq u \hookrightarrow w, w \hookrightarrow u \leq w \hookrightarrow v, u \otimes w \leq v\phi w, u \boxminus w \leq v \boxminus w, w \boxminus v \leq u \boxminus u$,
(4)$u \oplus v \leq u$ m $v \leq u, v \leq u \omega v \leq u \boxplus v$,

(5) $u \boxminus v \leq u, u \boxminus v \leq v$,

(6) $u \oplus u^{*} = 0, u \oplus 0 = 0$,

(7) $u \leq v$ iff $u \odot v^{*} = 0$ iff $u \boxminus v = 0$,

(8) $(u \boxminus v) \boxminus w = (u \boxminus w) \boxminus v$.

Definition 2 (see [24]). A nonempty subset I of an $R\ell$ -monoid $\mathcal{A}$ is an ideal if it satisfies the following conditions:

(1) if $u \leq v$ and $v \in I$, then $u \in I$,

(2) if $u, v \in I$, then $u \omega v \in I$.

Definition 3 (see [15]). A self-map d on an $R\ell$-monoid $\mathcal{M}$ is called a lattice derivation if it satisfies, for any $u, v \in \mathcal{M}$,

d(u ⋒ v) = (du ⋒ v) ⋓ u ⋒ dy). (6)

Definition 4 (see [24]). A self-map d on an $R\ell$-monoid $\mathcal{H}$ is called a multiplicative derivation if it satisfies, for any $u, v \in \mathcal{M}$,

$d(u \oplus v) = (\mathrm{d}u \odot v) w u \oplus \mathrm{d}v)$.(7)

Proposition 2 (see [22]). A self-map $d_a: \mathcal{M} \longrightarrow \mathcal{M}$

$d_a u = a \oplus u.$ (8)

On an $R\ell$-monoid $\mathcal{A}$ is a multiplicative derivation.

3. Subtractive Derivations of Rℓ-Monoids

Then, we introduce a new kind of derivations on Rℓ -monoids and give some characterizations of them.

Definition 5. Let $\mathcal{H}$ be an Rℓ-monoid. A mapping $d: L \hookrightarrow L$ is called a subtractive derivation on $\mathcal{U}$ if

$d(u \boxminus v) = (\mathrm{d}u \boxminus v) \oplus (u \boxminus \mathrm{d}v)$, (9)

for any $u, v \in \mathcal{M}$.

We will denote by $\mathcal{D}(\mathcal{H})$ to be the set of all subtractive derivations of $\mathcal{A}$.

Some examples of subtractive derivations on Rℓ-monoids are presented.

Example 1. Let $\mathcal{H}$ be an Rℓ-monoid. Define a mapping $0_d\ \mathcal{M} \longrightarrow \mathcal{M}$ by

$0_d(u) = 0,$ (10)

for all $u \in \mathcal{H}$. Then, $0_d \in \mathcal{D}(\mathcal{M})$. Moreover, defining $d_1: \mathcal{M} \longrightarrow \mathcal{M}$ by

$d_1(u) = u,$ (11)

for all $u \in \mathcal{M}$. Then, $d_1 \in \mathcal{D}(\mathcal{M})$.

Example 2. Let $\mathcal{M} = \{0, u, v, 1\}$ be a chain. Defining operations 9 and $\hookrightarrow$ as follows (see Table l):

Then, $(\mathcal{H}, \Theta, \hookrightarrow, \mathrm{n}, \boldsymbol{w}, 0,1)$ is an Rℓ-monoid. Now, we define $d: \mathcal{M} \longrightarrow \mathcal{M}$ as follows:

$$d(x) = \begin{cases} 0, & x = 0, u, \\ 1, & x = v, 1. \end{cases} \quad (12)$$

Then, $d \in \mathcal{D}(\mathcal{M})$.

TABLE 1: The operations of ⊛, and ↪.

⊛	0	u	v	1
0	0	0	0	0
u	0	u	u	u
v	0	u	u	v
1	0	u	v	1
↪	0	u	v	1
0	1	1	1	1
u	0	1	1	1
v	0	u	1	1
1	0	u	v	1

Example 3. Let *M*n be the standard n-valued *M*V-algebra, and hence an Rl-monoid, for some n ≥ 2.

$$d(u) = \begin{cases} \frac{1}{n-1}, & u = 1, \\ 0, & u \in \mathcal{M}_n - \{1\}. \end{cases} \qquad (13)$$

Then, $d \in \mathcal{D}(\mathcal{M})$.

Remark 1. Considering the subtractive derivation d in Example 3, we have $d(u \circledast v) = \mathrm{d}u = 0 \neq (\mathrm{d}u \circledast v)\omega(u \circledast \mathrm{d}v)$, which implies that d is not a multiplicative derivation on Moreover, d(u⋒v) = du ≠ 0⋓u = (du⋒v)⋓(u⋒dv), and $\mathcal{M}$. hence d not a lattice derivation. This all shows that not every subtractive derivation is a multiplicative or lattice derivation on M

Definition 6. A subtractive derivation d on an Rℓ-monoid $\mathcal{M}$ is called isotone if $u \leq v$ implies $\mathrm{d}u \leq \mathrm{d}v$ for any $u, v \in \mathcal{M}$.

Example 4. The subtractive derivations in Example 2, 3 are all isotone.

Proposition 3. If $d \in \mathcal{D}(\mathcal{M})$, then for any $u, v \in \mathcal{M}$,

(1) $\mathrm{d}0 = 0$,

(2) $\mathrm{d}u = du \circledast u$,

(3) $\mathrm{d}u \leq u$,

(4) d is isotone,

(5) $\mathrm{d}u \boxminus v \leq u \boxminus dv$,

(6) $\mathrm{d}u = d1 \circledast u \circledast (d(u^{**}))^{*}$,

(7) $d(u \boxminus v) \leq \mathrm{d}u \boxminus \mathrm{d}v \leq \mathrm{d}u \uplus \mathrm{d}v$.

Proof

(1) $\mathrm{d}0 = d(0 \boxminus 0) = (\mathrm{d}0 \boxminus 0)$® $(0 \boxminus \mathrm{d}0) = 0$.

(2) $\mathrm{d}u = d(u \boxminus 0) = (\mathrm{d}u \boxminus 0) \otimes (u \boxminus \mathrm{d}0) = \mathrm{d}u \circledast u$.

(3) $\mathrm{d}u = \mathrm{d}u \circledast u \leq 1 \circledast u = u$.

(4) If $u \leq v$, then $u = u$ 回 $v = v \circledast (v \hookrightarrow u)$, and hence

$$\begin{aligned} \mathrm{d}u &= d(u \Cap\mathrm{in}\; v) \\ &= d(v \circledast (vu)) \\ &= d(v \boxminus (v \hookrightarrow u)^{**}) \\ &= (\mathrm{d}v \boxminus (v \hookrightarrow u)^{**}) \circledast (v \boxminus d(v \hookrightarrow u)^{**}) \\ &\leq \mathrm{d}v \boxminus (v \hookrightarrow u)^{**} \\ &\leq \mathrm{d}v. \end{aligned} \qquad (14)$$

(5) It can be directly obtained from (2) and Proposition 1 (3).

(6) $\mathrm{d}u = d(1 \boxminus u^{**}) = (\mathrm{d}1 \circledast u) \circledast (d(u^{**})^{*}$.

(7) Obviously from Definition 5 and (3).

We will give some characterizations of subtractive derivations on $\mathcal{F}(\mathcal{M})$, which is a Gödel algebra, and study some of their basic algebraic properties.

Theorem 1. Let $d: \mathcal{M} \longrightarrow \mathcal{M}$ be a map on an $R\ell$-monoid $\mathcal{A}$. Then, the following are equivalent:

(1) $d \in \mathcal{D}(\mathcal{F}(\mathcal{M}))$,

(2) $d(u \boxminus v) = \mathrm{d}u \boxminus v, \forall u, v \in \mathcal{F}(\mathcal{M})$.

Proof
(1) ⇒ (2) if $d \in \mathcal{D}(\mathcal{F}(\mathcal{M}))$, then we have

$$\begin{aligned} d(u \boxminus v) &= (\mathrm{d}u \boxminus v) \circledast (u \boxminus \mathrm{d}v) \\ &\geq (\mathrm{d}u \boxminus v) \circledast (\mathrm{d}u \boxminus v) \qquad (15) \\ &= \mathrm{d}u \boxminus v. \end{aligned}$$

Conversely, $d(u \boxminus v) = (\mathrm{d}u \boxminus v) \circledast (u \boxminus \mathrm{d}v) \leq \mathrm{d}u \boxminus v$. So $d(u \boxminus v) = \mathrm{d}u \boxminus v, \forall u, v \in \mathcal{F}(\mathcal{M})$.

(2) ⇒ (1) let d be a map on $\mathcal{D}(\mathcal{F}(\mathcal{M}))$ such that $d(u \boxminus v) = \mathrm{d}u \boxminus v$, $\forall u, v \in \mathcal{F}(\mathcal{M})$. Then, $\mathrm{d}0 = d(0 \boxminus \mathrm{d}0) = \mathrm{d}0$ 日 $\mathrm{d}0 = 0$. Furthermore, $0 = d(u \boxminus u) = \mathrm{d}u \boxminus u$, which implies $\mathrm{d}u \leq u$, hence by Proposition 3 (6), we have $\mathrm{d}u \boxminus v \leq u$ 日 $\mathrm{d}v$. $\mathrm{d}u \boxminus v, \forall u, v \in \mathcal{F}(\mathcal{M})$.

Proposition 4. Let $d \in \mathcal{D}(\mathcal{M})$. Then, the following hold, $\forall u, v \in \mathcal{F}(\mathcal{M})$:

(1) $\mathrm{d}u = \mathrm{d}1 \circledast u = \mathrm{d}1$ 同 u,

(2) $d(u \circledast v) = \mathrm{d}u \circledast \mathrm{d}v$,

(3) $d(\mathcal{F}(\mathcal{M})) \subseteq \mathcal{F}(\mathcal{M})$,

(4) $d(u \uplus v) = \mathrm{d}u \uplus dv$,

(5) $d(u \hookrightarrow v) \leq \mathrm{d}u \hookrightarrow dv$,

(6) $u \in [0, d1]$ iff $\mathrm{d}u = u$,

(7) $\mathrm{d}1 \leq u$ iff $\mathrm{d}u = d1$.

Proof
(1) By Proposition 3(3), we have $u \leq (d(u^{**})^{*}$, and hence $\mathrm{d}u = \mathrm{d}1 \circledast u \circledast (d(u^{**})^{*} = \mathrm{d}i \circledast u \Cap (d(u^{**})^{*} = \mathrm{d}i * u = d1$ in u.

(2) By (1), we have $d(u \circledast v) = \mathrm{d}1$ ($(u \circledast v) = \mathrm{d}1 \Cap (u \cap v) = \mathrm{d}u$ ind v.

(3) If $u \in \mathcal{F}(\mathcal{M})$, then by (2), $d(u) = d(u \circledast u) = \mathrm{d}u \circledast \mathrm{d}u$, which shows $d(\mathcal{F}(\mathcal{M})) \subseteq \mathcal{F}(\mathcal{M})$.

(4) By (1), we have $d(u \uplus v) = \mathrm{d}1 \cap (u \uplus v) = (\mathrm{d}1 \cap u) \uplus (\mathrm{d}1\ \mathrm{m}v) = \mathrm{d}u \uplus \mathrm{d}v$.

(5) By (2), we have $\mathrm{d}u \circledast d(u \hookrightarrow v) = d(u \circledast (u \hookrightarrow v)) = d\ (u \text{ in } v) \leq \mathrm{d}v$, and hence $d(u \hookrightarrow v) \leq \mathrm{d}u \longrightarrow \mathrm{d}v$.

(6) and (7) are directly from (1), and hence we omit the proof of them.

4. The Fixed-Point Set of Subtractive Derivations on Rℓ-Monoids

Let $\mathcal{M}$ be an Rℓ-monoid. Define $F_{\mathcal{A}} = \{u \in \mathcal{M} \mid du = u\}$, which is called the fixed point set of subtractive derivation on an Rt-monoid $\mathcal{M}$.

Proposition 5. If $d \in \mathcal{D}(\mathcal{M})$, then $F_{\mathcal{A}} \subseteq \mathcal{F}(\mathcal{M})$.

Proof. If $u \in F_{\mathcal{H}}$, then by Proposition 3 (2), $\mathrm{d}u = \mathrm{d}u \circledast du$, and hence $u = u \circledast u$, which shows $u \in \mathcal{F}(\mathcal{M})$.

The converse of Proposition 5 is not true in general.

Example 5. Let $\mathcal{M} = \{0, u, v, 1\}$ be a chain. Defining operations $\circledast$ and $\hookrightarrow$ as follows (see Table 2):

Then, $(\mathcal{M}, \circledast, \hookrightarrow, \mathrm{f}, \boldsymbol{w}, 0,1)$ is an Rℓ-monoid. Defining d: $\mathcal{M} \longrightarrow \mathcal{U}$ as follows:

$$d(x) = \begin{cases} 0, & x = 0,1, \\ 1, & x = u, v. \end{cases} \qquad (16)$$

But $F_{\mathcal{H}} = \{0\} \subseteq \{0,1\} = \mathcal{F}(\mathcal{M})$ and $d \notin \mathcal{D}(\mathcal{M})$ since $d(u \boxminus v) = \mathrm{d}u = 1 \neq 0 = (\mathrm{d}u \boxminus v) \circledast (u \boxminus \mathrm{d}v)$.

Proposition 6. The identity map id $\mathrm{MU}_{\mathcal{B}} \in \mathcal{D}(\mathcal{M})$ iff $\mathcal{M}$ is a Gödel algebra.

Proof. If $id_{\mathcal{M}} \in \mathcal{D}(\mathcal{M})$, then by Proposition 5, $\mathcal{M} = F_{\mathcal{H}} \subseteq \mathcal{F}(\mathcal{M})$, and hence $\mathcal{M} = \mathcal{F}(\mathcal{M})$, which implies that $\mathcal{M}$ is a Gödel algebra.

Conversely, if $\mathcal{M}$ is a Gödel algebra, then $id_{\mathcal{M}} \in \mathcal{D}(\mathcal{M})$. Indeed, $id_{\mathcal{M}}(u \boxminus v) = u \boxminus v = id_{\mathcal{M}} u \boxminus v$, by Theorem 1 , id $_{\mathcal{U}} \in \mathcal{D}(\mathcal{M})$.

Proposition 6 shows that the identity map on a Gödel algebra is a subtractive derivation. Then, we give some conditions under which a subtractive derivation is identified.

TABLE 2: The operations of ⊛, and ↪.

⊛	0	u	v	1
0	0	0	0	0
u	0	u	u	u
v	0	0	u	v
1	0	u	v	1
↪	0	u	v	1
0	1	1	1	1
u	v	1	1	1
v	u	v	1	1
1	0	u	v	1

Theorem 2. Let $\mathcal{M}$ be a Gödel algebra and $d \in \mathcal{D}(\mathcal{M})$. Then, the following are equivalent:

(1) $d = id_{\mathcal{M}}$,

(2) $u \boxminus \mathrm{d}v = \mathrm{d}u \boxminus v$,

(3) d is injective.

Proof

(1) ⇒ (2) Obviously.

(2) ⇒ (1) if d satisfies $u \boxminus \mathrm{d}v = \mathrm{d}u \boxminus v$, then by Theorem 1 , $\mathrm{d}u = d(u \boxminus 0) = \mathrm{d}u \boxminus 0 = u \boxminus \mathrm{d}0 = u$, and hence $d = id_{\mathcal{M}}$.

(1) ⇒ (3) Obviously.

(3) ⇒ (1) if d is injective and for any $u \in \mathcal{M}$, then $d(u \boxminus u) = \mathrm{d}u \boxminus \mathrm{d}u = 0 = d0$, and hence $u \boxminus \mathrm{d}u = 0$, which implies $u \leq \mathrm{d}u$. So $\mathrm{d}u = u$ by Proposition 3 (3).

Proposition 7. Let $\mathcal{M}$ be a Gödel algebra and $d \in \mathcal{D}(\mathcal{M})$. Then

(1) if $u \in \mathcal{M}$ and $v \in F_{,}$, then $v \boxminus u \in F_{,\mu}$,

(2) if $v \in F_{\mathcal{M}}$ and $\forall u \in \mathcal{M}$, then v nin $u \in F_{\mathcal{H}}$.

Proof

(1) if $u \in \mathcal{M}$ and $v \in F_{\mathcal{H}}$, then $du = d$ and by Theorem 1, $d(v \boxminus u) = \mathrm{d}v \boxminus u = v \boxminus u$, which implies $v \boxminus u \in F_{\mathcal{H}}$.

(2) If $v \in F_{\mathcal{H}}$ and $\forall u \in \mathcal{M}$, then by Proposition 4 (2), $d($ ᴎ∩ $u) = \mathrm{d}v \circledast \mathrm{d}u = v$ vin u, which implies v in $u \in F_{\mu}$.

Proposition 8. Let $\mathcal{M}$ be an $R\ell$-monoid. Define a map $h_a : \mathcal{M} \longrightarrow \mathcal{M}, h_a u = u \boxminus a, \forall x, a \in \mathcal{M}$, then $h_a \in \mathcal{D}(\mathcal{M})$ iff $h_a(\mathcal{M}) \subseteq \mathcal{F}(\mathcal{M})$.

Proof. If $h_a(\mathcal{H}) \subseteq \mathcal{F}(\mathcal{M})$, then by Proposition 3 (3),

$$
\begin{aligned}
(h_a u \boxminus v) \circledast (u \boxminus h_a v) &= (h_a u \circledast u) \circledast (v^\star \oplus h_a v^\star) \\
&= (h_a u \oplus u) \oplus (v \uplus h_a v)^\star \\
&= h_a u \circledast v^\star \\
&= (u \boxminus a) \circledast v^* \qquad (17) \\
&= (u \boxminus a) \boxminus v \\
&= (u \boxminus v) \boxminus a \\
&= h_a(u \boxminus v),
\end{aligned}
$$

which implies $h_a \in \mathcal{D}(\mathcal{M})$.

Conversely, if $h_a \in \mathcal{D}(\mathcal{M})$, then

$$
\begin{aligned}
u \boxminus a &= h_a u \\
&= h_a(u \boxminus 0) \\
&= (h_a u \boxminus 0) \oplus (u \boxminus h_a 0) \qquad (18) \\
&= (u \boxminus a) \circledast (u \boxminus a), \forall u \in \mathcal{M},
\end{aligned}
$$

which implies

$h_a(\mathcal{M}) \subseteq \mathcal{F}(\mathcal{M})$.

Theorem 3. If $d \in \mathcal{D}(\mathcal{M})$ such that d is injective, then $F_{\mathcal{M}}$ is a lattice ideal iff $\mathcal{M}$ is a Gödel algebra.

Proof. If $d \in \mathcal{D}(\mathcal{M})$ such that d is injective and $\mathcal{U}$ is a Gödel algebra, then by Theorem 2(3), $d = id_{\mathcal{A}}$, and hence $F_{\mathcal{A}} = \mathcal{M}$, which shows that $F_{\mathcal{A}}$ is a lattice ideal.

Conversely, if $F_{\mathcal{A}}$ is a lattice ideal and $d \in \mathcal{D}(\mathcal{M})$ such that d is injective, then

$$
\begin{aligned}
d((d1)^\star) &= (1 \boxminus d1) \\
&= (d1 \boxminus d1) \circledast (1 \boxminus dd1) \qquad (19) \\
&= 0 \\
&= d0,
\end{aligned}
$$

that is $d1 = 1$, and hence $\mathcal{H} = F_{\mathcal{M}} \subseteq \mathcal{F}(\mathcal{M})$, which shows that $\mathcal{M}$ is a Gödel algebra.

Proposition 9. If $d_a(\mathcal{M}) \subseteq \mathcal{F}(\mathcal{M})$, then $d_a \in \mathcal{D}(\mathcal{M})$.

Proof. If $d_a(\mathcal{M}) \subseteq \mathcal{F}(\mathcal{M})$, then by Proposition 4 (3), $\forall u, v \in \mathcal{M}$,

$$
\begin{aligned}
(d_a u \boxminus v) \oplus (u \boxminus d_a v) &= (d_a u \oplus u) \oplus (v^* \oplus d_a v^*) \\
&= (d_a u \oplus u) \circledast (v \uplus d_a v)^* \\
&= d_a u \circledast v^* \\
&= (a \oplus u) \oplus v^* \qquad (20) \\
&= a \oplus (u \boxminus v) \\
&= d_a(u \boxminus v),
\end{aligned}
$$

which implies

$d_a \in \mathcal{D}(\mathcal{M})$.

Corollary 1. If $\mathcal{M}$ is a Gödel algebra, then $d_a \in \mathcal{D}(\mathcal{M})$.

Proposition 10. If $\mathcal{M}$ is a Gödel algebra, then the following hold:

(1) $d1 \in F_{\mathcal{A}}$,

(2) $d(\mathcal{M}) = F_{\mathcal{M}}$.

Proof

(1) It follows from Proposition 4 (1).

(2) It is obvious that $d(\mathcal{M}) \supseteq F_{\mathcal{A}}$. Conversely, if $u \in d(\mathcal{M})$, then there exists $v \in \mathcal{M}$ such that $u = \mathrm{d}v$. Since $u = dv \leq d1$ and $d1 \in F_{\mathcal{A}}$, by Theorem 2, $u \in F_{\mathcal{M}}$, and hence $d(\mathcal{M}) \subseteq F_{\mathcal{M}}$.

Theorem 4. If $\mathcal{M}$ is a Gödel algebra and I is a lattice ideal with the greatest element, then there exists $d \in \mathcal{D}(\mathcal{M})$ such that $F_{./} = I$.

Proof. If $b = \vee_{a\in I} a \in I$ and $d_b \in\in \mathcal{D}(\mathcal{M})$, then $d_b u \leq b$ with $b \in I$, and hence $d_b(\mathcal{H}) \subseteq I$. By Theorem 3, $d_b \in \mathcal{D}(\mathcal{M})$. Moreover, if $u \in I$, then $d_b u = umb$, and hence $u \in F_{.d}$ with respect to d_b, which implies $I \subseteq F_{\mathcal{M}}$. Furthermore, $F_{\mathcal{A}} = d(\mathcal{M})$, and hence $F_{\mathcal{A}} \subseteq I$ and $F_{\mathcal{A}} = I$.

5. The Relations between Kinds of Derivations on Rℓ-Monoids

In this section, we will discuss the relations between subtractive derivations and other derivations on Rℓ-monoids. In particular, we discuss the relations among subtractive derivations, lattice derivations, and multiplicative derivations on Rℓ-monoids.

Proposition 11. Every subtractive derivation is multiplicative on a Gödel algebra $\mathcal{M}$.

Proof. It follows from Propositions 4 (1) and (3).

Proposition 12. If d is a multiplicative derivation on an $R\ell$ -monoid $\mathcal{M}$ and $d(\mathcal{M}) \subseteq \mathcal{I}(\mathcal{M})$, then $d \in \mathcal{D}(\mathcal{M})$.

Proof. It follows from Propositions 2 and Corollary 1.

Proposition 13 (see [22]). If d is a multiplicative derivation on an $R\ell$-monoid $\mathcal{M}$ and $d1 \in \mathcal{F}(\mathcal{M})$, then the following are equivalent:

(1) d is isotone,

(2) $\mathrm{d}u = d1\%u$.

Proof. It follows from Propositions 4 (1) and (3).

Proposition 14 (see [22]). If d is a lattice derivation on an $R\ell$-monoid $\mathcal{M}$, then the following are equivalent:

(1) d is isotone;

(2) $\mathrm{d}u = d1\ \mathrm{m}u$.

Proof. It follows from Propositions 4 (1) and (3).

Theorem 5. If d is a map such that $dl \in \mathcal{I}(\mathcal{M})$ on an $R\ell$ -monoid $\mathcal{M}$, then d is a multiplicative derivation iff it is a lattice derivation.

Proof.

It follows from Propositions 13 and 14.

Proposition 15. Every subtractive derivation is multiplicative on a Gödel algebra M.

Proof.

It follows from Proposition 11.

Corollary 2. If d is a lattice derivation on an $R\ell$-monoid $\mathcal{M}$ and $d(\mathcal{M}) \subseteq \mathcal{F}(\mathcal{M})$, then $d \in \mathcal{D}(\mathcal{M})$.

Corollary 3. Subtractive derivations and lattice derivations are equivalent on the Gödel algebra.

6. Conclusions

The concept of subtractive derivations proves valuable when analyzing structures and characteristics within the realm of fuzzy logic algebra. To uncover shared traits among subtractive derivations in t-norm-based logical algebras, we introduce these derivations within RL-monoids and establish certain defining characteristics. Furthermore, we explore the connections between the fixed point set of subtractive derivations and other forms of derivations within RL-monoids. Looking ahead, our future work will center on investigating representations of RL-monoids using algebraic structures derived from the set of subtractive derivations.

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