Fourier Series Associate with Generalized Lauricella Function and the $\overline{\mathrm{H}}$-function

*R.P. Sharma

This paper presents some integral transformations and Fourier series involving a product of generalized Lauricella's functions and the $\overline{\mathrm{H}}$-function. The results established here are quite general and are capable of giving a number of new, interesting and useful integral transformations and Fourier series as its special cases.

Keywords : Fourier series, generalized Lauricella's functions, $\overline{\mathrm{H}}$-function.

Mathematics Subject Classification (2010): 33C45, 33C47, 33C60.

1. INTRODUCTION AND MAIN RESULTS

The $\overline{\mathrm{H}}$-function. introduced and defined [2] by Inayat-Hussian [5] and Lauricella"s functions [8], we derive the following results:

INTEGRAL TRANSFORMATIONS:

First Result:

$$\int_0^{\pi} (\sin x)^{w-1} e^{i\mu x} F^{E:F';\cdots;F^{(t)}}_{G:H';\cdots;H^{(t)}}\begin{pmatrix} y_1 (\sin x)^{2\lambda_1} \\ \vdots \\ y_t (\sin x)^{2\lambda_t}\end{pmatrix} \overline{H}\left(z(\sin x)^{2h}\right) dx$$

$$= \pi\, 2^{1-w} e^{\frac{i\mu x}{2}} \sum_{s_1;\cdots;s_t=0}^{\infty} \Delta\left\{\frac{(y_1 4^{-\lambda_1})^{s_1}}{s_1!}\cdots\frac{(y_t 4^{-\lambda_t})^{s_t}}{s_t!}\right\}$$

$$\overline{H}{}^{M,N+1}_{P+1,Q+2}\left[z4^{-h}\left|\begin{matrix}[1-w-2\lambda_1 s_1\cdots-2\lambda_t s_t, 2h;1], (a_j,\alpha_j;A_j)_{1,N}, (a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M}, (b_j,\beta_j;B_j)_{M+1,Q}, [\frac{1-w-2\lambda_1 s_1\cdots-2\lambda_t s_t\pm\mu}{2}, h;1]\end{matrix}\right.\right] \quad \ldots (1)$$

where $\mathrm{Re}(w)+2h \min_{1\le j\le M}[\mathrm{Re}(b_j/\beta_j)]>0,\ h>0, z>0, y_l>0, \lambda_l>0,$

$|\arg(z)|<\frac{\pi}{2}\Omega,\ \Omega>0,\ T_l'>0,\ l=1,\ldots,t,$ and the series on the right is convergent.

Second Result:

$$\int_0^{\pi}\cdots\int_0^{\pi}(\sin x_1)^{w_1-1}\cdots(\sin x_r)^{w_r-1}e^{i(\mu_1 x_1+\cdots+\mu_r x_r)}$$

$$F^{E_1:F_1';\cdots;F_1^{(t)}}_{G_1:H_1';\cdots;H_1^{(t)}}\begin{pmatrix} y_1'(\sin x_1)^{2\lambda_1'} \\ \vdots \\ y_t'(\sin x_1)^{2\lambda_t'}\end{pmatrix}\cdots F^{E_r:F_r';\cdots;F_r^{(t)}}_{G_r:H_r';\cdots;H_r^{(t)}}\begin{pmatrix} y_1^{(r)}(\sin x_r)^{2\lambda_1^{(r)}} \\ \vdots \\ y_t^{(r)}(\sin x_r)^{2\lambda_t^{(r)}}\end{pmatrix}$$

$$.\ \overline{H}\left(z(\sin x_1)^{2h'}\cdots(\sin x_r)^{2h^{(r)}}\right)dx_1\cdots dx_r$$

$$=\ \pi^r\ 2^{r-(w_1+\cdots+w_r)}\ e^{r\,(\mu_1+\cdots+\mu_r)\pi/2}\sum_{s_1';\ldots;s_t'=0}^{\infty}\cdots\sum_{s_1^{(r)};\cdots;s_t^{(r)}=0}^{\infty}(\Delta_1\cdots\Delta_r)$$

$$\left\{\frac{(y_1'4^{-\lambda_1'})^{s_1'}}{s_1'!}\cdots\frac{(y_t'4^{-\lambda_t'})^{s_t'}}{s_t'!}\right\}\cdots\left\{\frac{(y_1^{(r)}4^{-\lambda_1^{(r)}})^{s_1^{(r)}}}{s_1^{(r)}!}\cdots\frac{(y_t^{(r)}4^{-\lambda_t^{(r)}})^{s_t^{(r)}}}{s_t^{(r)}!}\right\}$$

$$\overline{H}^{M,N+r}_{P+r,Q+2r}\left[z\,4^{-(h'+\cdots+h^{(r)})}\left|\begin{matrix}[1-w_1-2\lambda_1's_1'\cdots-2\lambda_t's_t',2h';1];\ldots;\\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},\end{matrix}\right.\right.$$

$$\left.\begin{matrix}[1-w_r-2\lambda_1^{(r)}s_1^{(r)}\cdots-2\lambda_t^{(r)}s_t^{(r)},2h^{(r)};1],(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\ [\frac{1-w_1-2\lambda_1's_1'\cdots-2\lambda_t's_t'\pm\mu_1}{2},h';1];\ldots;[\frac{1-w_r-2\lambda_1^{(r)}s_1^{(r)}\cdots-2\lambda_t^{(r)}s_t^{(r)}\pm\mu_r}{2};h^{(r)};1]\end{matrix}\right]$$

… (2)

provided that

$$\mathrm{Re}(w_i)+(h',\ldots,h^{(r)})\min_{1\le j\le M}[\mathrm{Re}(b_j/\beta_j)>0,\ h',\ldots,h^{(r)}>0,\ y_1^{(i)},\ldots,y_t^{(i)}>0,\ \lambda_1^{(i)},\ldots,\lambda_t^{(i)}>0,$$

$z>0, |\arg(z)|<\frac{\pi}{2}\Omega,\ \Omega>0,\ T_l'>0,\ l=1,\ldots,t,$ and the series on the right is convergent.

FOURIER SERIERS: We establish the following Fourier series:

EXPONENTIAL FOURIER SERIERS

$$(2\sin x)^{w-1} F^{E:F';\cdots;F^{(t)}}_{G:H';\cdots;H^{(t)}}\begin{pmatrix} y_1(\sin x)^{2\lambda_1} \\ \vdots \\ y_t(\sin x)^{2\lambda_t}\end{pmatrix} \overline{H}\left(z(\sin x)^{2h}\right)$$

$$= \sum_{P=-\infty}^{\infty} \sum_{s_1;\cdots;s_t=0}^{\infty} \Delta e^{iP(\pi/2-x)} \left\{ \frac{(y_1 4^{-\lambda_1})^{s_1}}{s_1!} \cdots \frac{(y_t 4^{-\lambda_t})^{s_t}}{s_t!} \right\}$$

$$\overline{H}^{M,N+1}_{P+1,Q+2}\left[z4^{-h} \left| \begin{matrix} [1-w-2\lambda_1 s_1 \cdots -2\lambda_t s_t, 2h;1], (a_j,\alpha_j;A_j)_{1,N}, (a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M}, (b_j,\beta_j;B_j)_{M+1,Q}, [\frac{1-w-2\lambda_1 s_1 \cdots -2\lambda_t s_t \pm \mu}{2}, h;1] \end{matrix} \right. \right]$$

… (3)

valid under the conditions obtainable from (1).

SINE FOURIER SERIERS

$$(2\sin x)^{w-1} F^{E:F';\cdots;F^{(t)}}_{G:H';\cdots;H^{(t)}}\begin{pmatrix} y_1(\sin x)^{2\lambda_1} \\ \vdots \\ y_t(\sin x)^{2\lambda_t}\end{pmatrix} \overline{H}\left(z(\sin x)^{2h}\right)$$

$$= \sum_{P=-\infty}^{\infty} \sum_{s_1;\cdots;s_t=0}^{\infty} 2\Delta i^{-1} e^{iP\pi/2} \sin px \left\{ \frac{(y_1 4^{-\lambda_1})^{s_1}}{s_1!} \cdots \frac{(y_t 4^{-\lambda_t})^{s_t}}{s_t!} \right\}$$

$$\overline{H}^{M,N+1}_{P+1,Q+2}\left[z4^{-h} \left| \begin{matrix} [1-w-2\lambda_1 s_1 \cdots -2\lambda_t s_t, 2h;1], (a_j,\alpha_j;A_j)_{1,N}, (a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M}, (b_j,\beta_j;B_j)_{M+1,Q}, [\frac{1-w-2\lambda_1 s_1 \cdots -2\lambda_t s_t \pm \mu}{2}, h;1] \end{matrix} \right. \right]$$

… (4)

valid under the conditions obtainable from (1).

COSINE FOURIER SERIERS

$$(\sin x)^{w-1}\; F^{E:F';\cdots;F^{(t)}}_{G:H';\cdots;H^{(t)}}\begin{pmatrix} y_1(\sin x)^{2\lambda_1} \\ \vdots \\ y_t(\sin x)^{2\lambda_t}\end{pmatrix}\overline{H}\left(z(\sin x)^{2h}\right)$$

$$= 1/\sqrt{\pi}\sum_{s_1;\cdots;s_t=0}^{\infty}\Delta\left\{\frac{(y_1 4^{-\lambda_1})^{s_1}}{s_1\,!}\cdots\frac{(y_t 4^{-\lambda_t})^{s_t}}{s_t\,!}\right\}$$

$$\overline{H}^{M,N+1}_{P+1,Q+1}\left[z\left|\begin{matrix}[\frac{2-w}{2}-\lambda_1 s_1\cdots-\lambda_t s_t,h;1],(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},[\frac{1-w}{2}-\lambda_1 s_1\cdots-\lambda_t s_t,\,h\,;1]\end{matrix}\right.\right]$$

$$+\sum_{P=1}^{\infty}\sum_{s_1;\cdots;s_t=0}^{\infty}\Delta\, 2^{2-w}\, e^{iP\pi/2}\cos px\left\{\frac{(y_1 4^{-\lambda_1})^{s_1}}{s_1\,!}\cdots\frac{(y_t 4^{-\lambda_t})^{s_t}}{s_t\,!}\right\}$$

$$\overline{H}^{M,N+1}_{P+1,Q+2}\left[z\,4^{-h}\left|\begin{matrix}[1-w-2\lambda_1 s_1\cdots-2\lambda_t s_t,2h;1],(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},[\frac{1-w-2\lambda_1 s_1\cdots-2\lambda_t s_t\pm\mu}{2},\,h\,;1]\end{matrix}\right.\right]$$

$$\ldots (5)$$

valid under the conditions obtainable from (1).

MULTIPLE EXPONENTIAL FOURIER SERIES

$$(\sin x_1)^{w_1-1}\cdots(\sin x_r)^{w_r-1}$$

$$F^{E_1:F_1';\cdots;F_1^{(t)}}_{G_1:H_1';\cdots;H_1^{(t)}}\begin{pmatrix} y_1'(\sin x_1)^{2\lambda_1'} \\ \vdots \\ y_t'(\sin x_1)^{2\lambda_t'}\end{pmatrix}\cdots F^{E_r:F_r';\cdots;F_r^{(t)}}_{G_r:H_r';\cdots;H_r^{(t)}}\begin{pmatrix} y_1^{(r)}(\sin x_r)^{2\lambda_1^{(r)}} \\ \vdots \\ y_t^{(r)}(\sin x_r)^{2\lambda_t^{(r)}}\end{pmatrix}$$

$$.\quad \overline{H}\left(z(\sin x_1)^{2h'}\cdots(\sin x_r)^{2h^{(r)}}\right)$$

$$=\sum_{P_1,\ldots,P_r=-\infty}^{\infty}\;\sum_{s_1',\ldots,s_t'=0}^{\infty}\cdots\sum_{s_1^{(r)},\ldots,s_t^{(r)}=0}^{\infty}(\Delta_1\cdots\Delta_r)$$

$$2^{r-(w_1+\cdots+w_r)}\ e^{-i(p_1x_1+\cdots+p_rx_r)}\ e^{\ i(p_1+\cdots+p_r)\pi/2}$$

$$\left\{\frac{(y_1'\,4^{-\lambda_1'})^{s_1'}}{s_1'\,!}\cdots\frac{(y_t'\,4^{-\lambda_t'})^{s_t'}}{s_t'\,!}\right\}\cdots\left\{\frac{(y_1^{(r)}\,4^{-\lambda_1^{(r)}})^{s_1^{(r)}}}{s_1^{(r)}\,!}\cdots\frac{(y_t^{(r)}\,4^{-\lambda_t^{(r)}})^{s_t^{(r)}}}{s_t^{(r)}\,!}\right\}$$

$$\bar{H}^{M,N+r}_{P+r,Q+2r}\left[z\,4^{-(h'+\cdots+h^{(r)})}\left|\begin{matrix}[1-w_1-2\lambda_1's_1'\cdots-2\lambda_t's_t',2h';1];\ldots;\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},\end{matrix}\right.\right.$$

$$\left.\begin{matrix}[1-w_r-2\lambda_1^{(r)}s_1^{(r)}\cdots-2\lambda_t^{(r)}s_t^{(r)},2h^{(r)};1],(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\ [\frac{1-w_1-2\lambda_1's_1'\cdots-2\lambda_t's_t'\pm\mu_1}{2},h';1];\ldots;[\frac{1-w_r-2\lambda_1^{(r)}s_1^{(r)}\cdots-2\lambda_t^{(r)}s_t^{(r)}\pm\mu_r}{2};h^{(r)};1]\end{matrix}\right]$$

… (6)

valid under the conditions obtainable from (2).

PROOFS

The result (1) can be established by making use of a known result ([6], eqn.3.1.5, p.70 and [3]) and the result (2) is a straightforward generalization of (1).

To prove (3), let

$$f(x)=(\sin x)^{w-1}\,F^{E:F';\cdots;F^{(t)}}_{G:H';\cdots;H^{(t)}}\begin{pmatrix}y_1(\sin x)^{2\lambda_1}\\ \vdots\\ y_t(\sin x)^{2\lambda_t}\end{pmatrix}\ \bar{H}\left(z(\sin x)^{2h}\right)$$

$$=\sum_{p=-\infty}^{\infty}A_p\,e^{-ipx} \qquad \ldots (7)$$

Now, multiplying both sides of (7) by $e^{i\mu x}$ and integrating it with respect to x from 0 to π and changing the order of integration and summation (which is easily permissible) on the right, using the orthogonal relationship for exponential functions

$$\int_0^{\pi} e^{i(m-n)\pi}\, dx = \begin{cases} \pi, & m = n \\ 0, & m \neq n \end{cases} \qquad \dots (8)$$

on the right hand side and the result (1) on the left hand side of (7). We obtain the value of A_P, substituting this value of A_P in (7). Thus, we obtain the required exponential Fourier series in (3).

Similarly the other Fourier series expansion formulae from (4) and (5) can be obtained easily by making use of sine functions and cosine functions.

The proof of (6) can be developed on similar lines.

2. SPECIAL CASES

(i) For t=2, the integrals in (1), (2) and the Fourier series in (3) reduces to the following results involving the generalized *Kampẽ de Fẽriẽt* function ([7],p.199,see also [8],p.450).

$$\int_0^{\pi} (\sin x)^{w-1} e^{i\mu x} S^{E:F';F''}_{G:H';H''}\begin{pmatrix} y_1 (\sin x)^{2\lambda_1} \\ y_2 (\sin x)^{2\lambda_2} \end{pmatrix} \overline{H}\left(z (\sin x)^{2h}\right) dx$$

$$= \pi\, 2^{1-w}\, e^{\frac{i\mu x}{2}} \sum_{s_1, s_2 = 0}^{\infty} Q \left\{ \frac{(y_1 4^{-\lambda_1})^{s_1}}{s_1\,!} \frac{(y_2 4^{-\lambda_2})^{s_2}}{s_2\,!} \right\}$$

$$\overline{H}^{M,N+1}_{P+1,Q+2}\left[z 4^{-h} \left| \begin{matrix} [1-w-2\lambda_1 s_1 - 2\lambda_2 s_2, 2h; 1], (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q}, [\frac{1-w-2\lambda_1 s_1 - 2\lambda_2 s_2 \pm \mu}{2}, h; 1] \end{matrix} \right. \right]$$

$$\dots (9)$$

where $\mathrm{Re}(w) + 2h \min_{1 \le j \le M} [\mathrm{Re}(b_j / \beta_j)] > 0,\ y_1 > 0, y_2 > 0, \lambda_1 > 0, \lambda_2 > 0,, h > 0, z > 0,$

$$|\arg(z)| < \frac{\pi}{2}\Omega,\ \Omega > 0,\ T_1' > 0, T_2' > 0,\ l = 1, \dots, t, j = 1, \dots, u^{(i)}$$

Here $T_1' = 1 + \sum_{k=1}^{G} \beta_k' + \sum_{k=1}^{H'} \eta_k' - \sum_{k=1}^{E} \alpha_k' - \sum_{k=1}^{F'} \gamma_k' > 0,$

$$T_2' = 1 + \sum_{k=1}^{G} \beta_k'' + \sum_{k=1}^{H''} \eta_k'' - \sum_{k=1}^{E} \alpha_k'' - \sum_{k=1}^{F''} \gamma_k'' > 0.$$

$$\int_0^{\pi} \cdots \int_0^{\pi} (\sin x_1)^{w_1 - 1} \cdots (\sin x_r)^{w_r - 1} e^{i(\mu_1 x_1 + \cdots + \mu_r x_r)}$$

$$S_{G_1 : H_1' ; H_1''}^{E_1 : F_1' ; F_1''} \begin{pmatrix} y_1' (\sin x_1)^{2\lambda_1'} : \\ y_2' (\sin x_1)^{2\lambda_2'} \end{pmatrix} \cdots S_{G_r : H_r' ; H_r''}^{E_r : F_r' ; ; F_r''} \begin{pmatrix} y_1^{(r)} (\sin x_r)^{2\lambda_1^{(r)}} \\ y_2^{(r)} (\sin x_r)^{2\lambda_2^{(r)}} \end{pmatrix}$$

$$. \overline{H}\left(z (\sin x_1)^{2h'} \cdots (\sin x_r)^{2h^{(r)}}\right) dx_1 \cdots dx_r$$

$$= \pi^r \, 2^{r-(w_1+\cdots+w_r)} \, e^{i(\mu_1+\cdots+\mu_r)\pi/2} \sum_{s_1', s_2'=0}^{\infty} \cdots \sum_{s_1^{(r)}, s_2^{(r)}=0}^{\infty} (Q_1 \cdots Q_r)$$

$$\left\{ \frac{(y_1' 4^{-\lambda_1'})^{s_1'}}{s_1' !} \frac{(y_2' 4^{-\lambda_2'})^{s_2'}}{s_2' !} \right\} \cdots \left\{ \frac{(y_1^{(r)} 4^{-\lambda_1^{(r)}})^{s_1^{(r)}}}{s_1^{(r)} !} \frac{(y_2^{(r)} 4^{-\lambda_2^{(r)}})^{s_2^{(r)}}}{s_2^{(r)} !} \right\}$$

$$\overline{H}_{P+r, Q+2r}^{M, N+r} \left[z 4^{-(h' + \cdots + h^{(r)})} \left| \begin{matrix} [1 - w_1 - 2\lambda_1' s_1' - 2\lambda_2' s_2', 2h'; 1]; \ldots ; \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q}, \end{matrix} \right. \right.$$

$$\left. \begin{matrix} [1 - w_r - 2\lambda_1^{(r)} s_1^{(r)} - 2\lambda_2^{(r)} s_2^{(r)}, 2h^{(r)}; 1], (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ [\frac{1 - w_1 - 2\lambda_1' s_1' - 2\lambda_2' s_2' \pm \mu_1}{2}, h'; 1]; \ldots ; [\frac{1 - w_r - 2\lambda_1^{(r)} s_1^{(r)} - 2\lambda_2^{(r)} s_2^{(r)} \pm \mu_r}{2}; h^{(r)}; 1] \end{matrix} \right]$$

... (10)

provided that

$$\mathrm{Re}(w_i) + (h', \ldots, h^r) \min_{1 \le j \le M} [\mathrm{Re}(b_j / \beta_j) > 0, \; h', \ldots, h^{(r)} > 0, y_1^{(i)} > 0, y_2^{(i)} > 0,$$

$$\lambda_1^{(i)} > 0, \lambda_2^{(i)} > 0, z > 0, |\arg(z)| < \frac{\pi}{2} \Omega, \; \Omega > 0, T_{1i}' > 0, T_{2i}' > 0, \; i = 1, \ldots, r, \; j = 1, \ldots, u^{(i)}$$

EXPONENTIAL FOURIER SERIERS

$$(2\sin x)^{w-1} S^{E:F';F''}_{G:H';H''}\begin{pmatrix} y_1 (\sin x)^{2\lambda_1} \\ y_2 (\sin x)^{2\lambda_2} \end{pmatrix} \overline{H}\left(z (\sin x)^{2h}\right)$$

$$= \sum_{P=-\infty}^{\infty} \sum_{s_1;s_2=0}^{\infty} Q e^{iP(\pi/2-x)} \left\{ \frac{(y_1 4^{-\lambda_1})^{s_1}}{s_1 !} \frac{(y_2 4^{-\lambda_2})^{s_2}}{s_2 !} \right\} \quad \ldots (11)$$

valid under the conditions obtainable from (9). Similarly the other Fourier series (Sine Fourier series, Cosine Fourier series and Multiple Exponential Fourier series) can be reduced to the results involving the generalized *Kampẽ de Fẽriẽt* functions [1]and H-function [4]

***Govt. Engineering College**
Ajmer (Raj)

REFERENCES

[1] Appell,P.et J. *Kampẽ de Fẽriẽt*, Functions Hypergemetriques et Hyperspheriques, polynomes d'Hermite, Gauthier-Villars, Paris,1926.

[2] Buschman, R.G. and Srivastava, H.M.(1990). The $\overline{H}$ -function associated with a certain class of Feynman integrals, J. Phys. A : Math. Gen., **23**), 4707-4710.

[3] Chandel R.C.Singh, Agrawal R.D. and Hemant kumer, Fourier series involving the multivariable H-function of Srivastava and Panda, Indian J.Pure Appl. Math.23(5) (1992),343-357.

[4] Fox, C. (1961). The G and H-functions as symmetrical Fourier kernels, Trans. Amer . Math.Soc., **98**, 395-429.

[5] Inayat-Hussain, A.A. (1987). New properties of hypergeometric series derivable from Feynman integrals: II. A generalization of the H-function, J. Phys. A : Math. Gen., **20**, 4119-4128.

[6] Mathai A.M.and.Saxena R,K, Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences, Springer-Verlag, New York,1973.

[7] Srivastava H.M.and Martha C.Daoust, on Eulerian Integrals associated with kampe de feriet functions.Publ.Inst.Math. (Beograd) nouvellr Ser. 9(23) (1969),199-202.

[8] Srivastava H.M.and Martha C.Daoust, Certain generalized Neumann expansions associated with the *Kampẽ de Fẽriẽt* function, Nederl.Akad.Wetensch.Proc.Ser.A72, Indag.Math.31 (1969),449-457.