A Study on Isomorphism of Partitioning Hamiltonian Circuits in Complete Graphs

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Abstract

This study introduces a novel approach for partitioning isomorphic Hamiltonian circuits in full graphs. The isomorphism classes will be constructed using matrix theory concepts. This technique focuses on categorizing Hamiltonian circuits using matrix transposition theory. This idea will be shown by decomposing and partitioning a full graph of order five. This technique develops and proves a fresh outcome.

Keywords: Decomposition, Complete Graphs, Isomorphism, Hamiltonian Circuits.

Introduction

An isomorphism of a graph is an overwhelmingly interesting problem due to the fact that it can be adopted in the field of organic chemistry to determine two identical molecules (Milan (1977), Jean-Loup (1998)). In graph theory, a complete graph $K$is known to have n! Hamiltonian circuits (HC) and $(n-1)!/\ 2$ distinct HC (Riaz and Khiyal (2006), Douglas (2001)). The isomorphic HC among those n! circuits need to be classified to produce the distinct HC. Thus, in this paper, we aim to construct a cutting-edge method to partition isomorphic HC in Kn. The idea of adjacency matrix and matrix transposition are considered in order to classify the isomorphism classes of the HC. The definitions needed along this paper are given below.

Definition 1 A complete graph Kn is a simple graph with n vertices whose vertices are pairwise adjacent.

Definition 2 A Hamiltonian circuit is a circuit that starts and ends at the same vertex, and visits each vertex exactly once.

Definition 3 Let $C1* = (V1, E1)$ and $C2* = (V2, E2)$ be two Hamiltonian circuits. $C1* \cong C2*$ if there is a one-to-one function f: $V(C1*) \rightarrow V(C2*)$ such that $uv \in E(C1*)$ if and only if $f(u)f(v) \in (C2*)$.

Definition 4 Suppose $G = (V, E)$ where $v1, v2, v3, \dots, vn \in V$. The adjacency matrix A of G (or AG), with respect to this listing of vertices, is the $n \times n$ matrix with 1 as its (i,j)th entry when vi and vj are adjacent, 0 as its (i,j)th entry when they are not adjacent.

Definition 5 If M is a $m \times n$ matrix, then the transpose matrix of M denoted by MT is an $n \times m$ matrix, where the columns of M be the rows of MT and the rows of M be the columns of MT .

In linear algebra, a symmetric matrix is a square matrix that is equal to its transpose. Formally, a matrix M is symmetric if M = MT.

Definition 6 Let $C1*$ be a circuit with direction ($x1$, $x2$, $x3$, … , xn−1, xn, $x1$). Then, a circuit $C2*$ is a mirror image to circuit $C1*$ if the direction of $C2*$ is ($x1$, xn, xn−1, … , $x3$, $x2$, $x1$).

Definition 7 Let $C1*$ and $C2*$ be two circuits with n vertices. If $C2*$ is the mirror image of $C1*$, then $C1* \cong C2*$.

Definition 8 If the mapping of $C1*$ and $C2*$ is (1, a)(2, b)(3, c) … (n, z) and (z, n) … (c, 3)(b, 2)(a, 1) respectively, then $C1*$ and $C2*$ has an opposite mapping, where a, b, c, … , z are the images.

Definition 9 Suppose the sets of vertices $\{x1, x2, x3, \ldots, xn\} \in C1*$ and $\{x1a, x2b, x3c, \ldots, xnz\} \in C2*$. A function g = ($x1$ $x1a$ $x2$ $x2b$ $x3$ $x3c$ … … xn xnz) maps the vertices $\{x1, x2, x3, \ldots , xn\}$ of $C1*$ to other vertices $\{x1a, x2b, x3c, \ldots , xnz\}$ of $C2*$ where $\{x1a, x2b, x3c, \ldots , xnz\}$ are the images. That is, $x1 \longmapsto x2b$, $x2 \longmapsto x3c$, …, $xn \longmapsto xnz$ for $n \in \mathbb{Z}+$. Then, the mapping is written as a product of transposition ($x1$, $x1a$)($x2$, $x2b$) … (xn, xnz).

Methodology

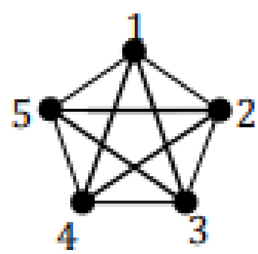

Figure 1: A complete graph, $K5$.

Now we consider $K5$ as shown in Figure 1, it can be decomposed into (5 − 1)! = 24 HC from $K5$ (Riyaz and Khiyal,2006). Since there are (n−1)! 2 distinct HC in $K5$ (Douglas,2001), we use the idea of adjacency matrix and transpose matrix to partition the isomorphism HC to get the distinct HC. Among the twenty four circuits, as an example, we provide several HC from $K5$ in Figure 2. Then, the adjacency matrix as well as its transpose are presented in Table 1. Then, we investigate which matrices are symmetric.

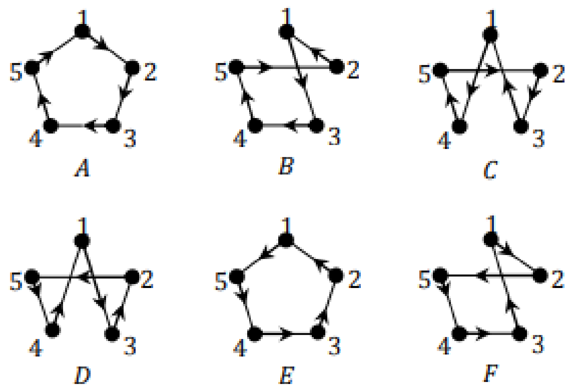

Figure 2: Several HC from $K5$.

Table 1: Adjacency matrix and its transpose for $K5$

Adjacency matrix	The transpose
$A = \begin{array}{c\|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 0 & 1 \\ 5 & 1 & 0 & 0 & 0 & 0 \end{array}$	$A^{T} = \begin{array}{c\|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 1 & 0 & 0 \\ 5 & 0 & 0 & 0 & 1 & 0 \end{array}$
$B = \begin{array}{c\|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 0 & 1 \\ 5 & 0 & 1 & 0 & 0 & 0 \end{array}$	$B^{T} = \begin{array}{c\|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 1 & 0 & 0 \\ 5 & 0 & 0 & 0 & 1 & 0 \end{array}$

$C = \begin{array}{c\|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 1 \\ 5 & 0 & 1 & 0 & 0 & 0 \end{array}$	$C^{\mathrm{T}} = \begin{array}{c\|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \\ 3 & 0 & 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 1 & 0 \end{array}$
$D = \begin{array}{c\|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \\ 3 & 0 & 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 1 & 0 \end{array}$	$D^{\mathrm{T}} = \begin{array}{c\|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 1 \\ 5 & 0 & 1 & 0 & 0 & 0 \end{array}$
$E = \begin{array}{c\|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 1 & 0 & 0 \\ 5 & 0 & 0 & 0 & 1 & 0 \end{array}$	$E^{\mathrm{T}} = \begin{array}{c\|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 0 & 1 \\ 5 & 1 & 0 & 0 & 0 & 0 \end{array}$
$F = \begin{array}{c\|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 1 & 0 & 0 \\ 5 & 0 & 0 & 0 & 1 & 0 \end{array}$	$F^{\mathrm{T}} = \begin{array}{c\|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 0 & 1 \\ 5 & 0 & 1 & 0 & 0 & 0 \end{array}$

There are symmetric matrices produced in Table 1, i.e. $A\ T = E$, $B\ T = F$, $C\ T = D$, $D\ T = C$, $E\ T = A$, and $F\ T = B$. Without loss of generality, since $K5$ has twenty four HC, thus $K5$ can be partitioned into twelve distinct HC

RESULT

From the case $K5$ discussed in previous section, a theorem is produced as shown in the next paragraph.

A complete graph Kn is known can be partitioned into $(n-1)!$ HC (Riyaz and Khiyal,2006). Suppose a HC as shown in Figure 3. To partition the isomorphic classes of the circuits, we use the idea of adjacency matrices and transpose matrices as discussed below.

Step 1: Find the adjacency matrices of each HC.

Step 2: Find the transpose matrices of each adjacency matrix obtained in Step 1.

Step 3: Investigate which matrices are symmetric to partition the isomorphic circuits.

Step 4: Determine the distinct HC. Without loss of generality, we have the following theorem

Theorem 1. Let P and Q be two Hamiltonian circuits with opposite direction. If the adjacency matrix of P equals to the transpose of adjacency matrix of Q (P = Q T), then circuit $P \cong$ �

Proof. Suppose P and Q are two Hamiltonian circuits with n vertices as shown in Figure 3.

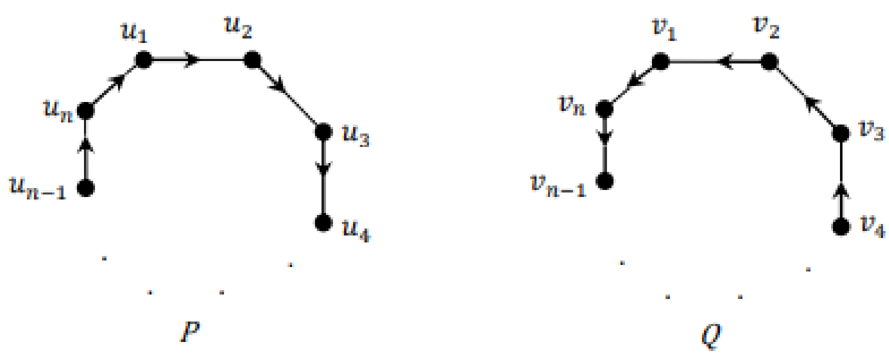

Figure 3: A complete graph Kn

Both P and Q have n vertices, n edges, and vertices of degree two. Because P and Q agree with respect to these invariants, we define a function f to investigate the one-to-one function. Since all vertices in both P and Q have degree two, then we have $f(un) = v3$, $f(u1) = v2$, $f(u2) = v1$, $f(u3) = vn$, $f(u4) = vn-1$, …, $f(un-1) = v4$. To examine whether f preserves edges, we examine the adjacency matrices of P and Q as well as their transpose, with the rows and columns labeled by the images of their corresponding vertices.

$$\mathbf{P} = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_{n-1} \\ u_n \end{array} \begin{array}{c} \begin{array}{ccccccc} u_1 & u_2 & u_3 & u_4 & \dots & u_{n-1} & u_n \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 1 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & & 0 & 1 \\ 1 & 0 & 0 & 0 & & 0 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{P}^T = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_{n-1} \\ u_n \end{array} \begin{array}{c} \begin{array}{ccccccc} u_1 & u_2 & u_3 & u_4 & \dots & u_{n-1} & u_n \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & & 0 & 1 \\ 1 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 0 & & 1 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{Q} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ \vdots \\ v_{n-1} \\ v_n \end{array} \begin{array}{c} \begin{array}{ccccccc} v_1 & v_2 & v_3 & v_4 & \dots & v_{n-1} & v_n \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & & 0 & 1 \\ 1 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 0 & & 1 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{Q}^T = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ \vdots \\ v_{n-1} \\ v_n \end{array} \begin{array}{c} \begin{array}{ccccccc} v_1 & v_2 & v_3 & v_4 & \dots & v_{n-1} & v_n \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 1 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & & 0 & 1 \\ 1 & 0 & 0 & 0 & & 0 & 0 \end{bmatrix} \end{array}$$

From the above matrices, we have adjacency matrices P = Q T and Q = P T which shows that f preserves the edges. Thus, we conclude that P and Q are isomorphic.

DISCUSSIONS

We have developed a new approach in partitioning the isomorphic classes of HC in Kn. A case of n = 5 is discussed as a basis to find the isomorphism among the HC. A theorem has been produced to prove that two circuits are isomorphic if both circuits share the same edges.

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