''Investigating the use of Special Functions in Solving Fractional Differential Equations''

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ABSTRACT

Fractional differential equations are generalized forms of classical differential equation that incorporate non-locality and memory effects. They can be derived in both space and time using power-law kernels, which account for hereditary and memory-dependent behaviors of various substances and processes. As powerful mathematical tools, fractional differential equations provide a rigorous framework for modelling complex dynamical systems where traditional approaches are insufficient. Their well-established physical foundation has opened avenues for new scientific investigations, including theoretical analysis, numerical methods, and applications across diverse fields such as physics, engineering, and applied mathematics.

The use of special functions has recently expanded to include the solution of stochastic, integro-differential, and fractional differential equations that arise in the mathematical modelling of real-world problems in science and engineering. These functions are particularly valuable in solving partial differential equations, where variable separation often reduces them to systems of ordinary differential equations. Among the most widely used special functions are hypergeometric functions, Hermite polynomials, Zeta functions, Bessel functions, Legendre polynomials, Laguerre polynomials, and Chebyshev polynomial. The theory of orthogonal polynomials plays a central role in obtaining precise and accurate solutions to differential equations. Their orthogonality with respect to specific weight functions and parameter choices is essential for investigating accuracy, stability, and approximate solutions to time-dependent PDEs.

KEYWORDS: Fractional Calculus, Fractional Differential Equation, Special Function, Mittag-Leffler Function, Gamma Function, Beta Function.

INTRODUCTION

Fractional differential equations (FDEs) works as a important tool for modeling complex processes in science and engineering. FDEs allow derivatives of any real or fractional order. This is the reason makes them useful for describing memory, hereditary properties , and irregular behaviors that appear in areas such as physics , biology , control theory , and finance.

By solving these equations, mathematicians often make use of special function. Special functions play a key role because they naturally appear in the solution of fractional models. The idea of fractional calculus, which offers with derivatives and integrals of arbitrary orders.

There are different approaches for defining fractional derivatives. More specifically, an integral and a derivative operators based on the other operator such as Fourier transform, Laplace transform, n-fold integral operator, finite difference and so on can be extended to upper dimensions. That is, let

$f_{(n)}$(x)=$\int_a^x \int_a^{x_1} \ldots \int_x^{x_{n-1}} f(x-1)dx_{n-1}$. . .d$x_1$dx

=$\frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}$f(t)dt,

$f^{(n)}(x) = \frac{d^n}{dx^n} f(x)$,

where$f_{(n)}(x)$ and $f^{(n)}(x)$ are respectively an n-fold integral and a derivatives of order n. There are different approaches for presenting the functions $f_{(n)}(x)$ and $f^{(n)}(x)$ in terms of f(x) and n, each approach leading to a definition for the fractional derivative.

Riemann-Liouville Fractional Derivative

We are starting here, with the definition for the Riemann-Liouville (RL) fractional integrals and derivatives. With this definition the RL fractional derivatives is discussed. This comes from an n-fold integral operator.

Let I= [a, b] be a finite interval on the real axis R. Fractional integrals $RL_{I_{a+}^{\alpha}} f$ and $RL_{I_{b-}^{\alpha}} f$ of order $\alpha \in \mathbb{C}$ are defined by

$(RL_{I_{a+}^{\alpha}} f)$ (x)=$\frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}}$,(x> a ;Re$(\alpha) > 0)$ (1.1)

and ($RL_{I_{b-}^{\alpha}} f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \frac{f(t)dt}{(t-x)^{1-\alpha}}$, $(x < a; Re(\alpha) > 0)$ (1.2)

where $\Gamma(x)$ and Re(α) show the Gamma function and Imaginary part of α, respectively. By setting n$\in N$ in equation (1.1) & (1.2) ,we obtain

$\left(RL_{I_{a+}^{\alpha}} f\right)(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)dt$, $(n \in N)$ (1.3)

nd $(RL_{I_{b-}^{\alpha}} f)$ $(x) = \frac{1}{(n-1)!}\int_x^b (t-x)^{n-1} f(t)dt$, $(n \in N)$

fractional derivatives $RL_{D_{a+}^{\alpha}} f$ *and* $RL_{D_{b-}^{\alpha}} f$ or order $\alpha \in C$ are defined by

$RL_{D_{a+}^{\alpha}} f)$ $(x) = (\frac{d}{dx})^n \left(RL_{I_{a+}^{n-\alpha}} f\right)(x)$

$= \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dx})^n \int_a^x \frac{f(t)dt}{(x-t)^{\alpha-n+1}}$, $(n = [Re(\alpha)] = 1; x > a)$

and $(\left(RL_{D_{b-}^{\alpha}} f\right)(x) = (-\frac{d}{dx})^n \left(RL_{I_{b-}^{n-\alpha}} f\right)(x)$

$= \frac{1}{\Gamma(n-\alpha)} (-\frac{d}{dx})^n \int_x^b \frac{f(t)dt}{(t-x)^{\alpha-n+1}}$, $(n = [Re(\alpha)]$+1 ; x< $b)$

Where [Re (α)] shows the integer part of Re(α)

Let n$\in N$, then $(RL_{D_{a+}^{0}} f)$ $(x) = \left(RL_{D_{b-}^{0}} f\right)(x) = f(x)$,

$\left(RL_{D_{a+}^{n}} f\right)(x) = f^{(n)}(x)$

and $(RL_{D^{0}_{b-}} f)\ (x) = (-1)^n f^{(n)}(x)\ \ , n \notin N$

Therefore, one important conclusion is that the RL fractional derivative of a constant number is not zero. This property in fractional calculus is in contrast to the property in ordinary calculus.

Where the derivative of a constant function is always zero.

CAPUTO FRACTIONAL DERIVATIVE

The Caputo fractional derivative is defined based on the RL fractional derivative and the Taylor series expansion. If the order of the derivative belongs to N, then the RL and C fractional derivative are equal.

Let I= [a, b] be a finite interval on the real line R, $\alpha \in C\ (Re(\alpha) \geq 0)$ and n=[Re(α)] + 1

The Caputo (C) fractional derivative are defined by

$$(c_{D^{\alpha}_{a+}} f)\ (x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}}{(x-t)^{\alpha-n+1\alpha}} (t)dt = (RL_{I^{n-\alpha}_{a+}} (\frac{d}{dx})^n f)(\text{x}),$$

And $(c_{D^{\alpha}_{b-}} f)(x) = \frac{1}{\Gamma(n-\alpha)} \int_x^b \frac{f^{(n)}}{(t-x)^{\alpha-n+1}} (t)dt = (-1)^n (RL_{I^{n-\alpha}_{a+}} \left(\frac{d}{dx}\right)^n f\Big) (x)$, respectively, when 0< $Re(\alpha) < 1$

Methods for solving Fractional Differential Equation:

ANALYTICAL METHOD: Fourier Transform Method

The Fourier Transform is a useful tool for solving many differential equation.

Example:- Consider a damped harmonic oscillator subjected to an additional driving force f(t). This force has an arbitrary time dependence, and is not necessarily harmonic. The equation of motion is

$$\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x(t) = \frac{f(t)}{m}$$

To solve for $x(t)$,first take the Fourier transform.

The result is

$$-\omega^2\ \text{X}\ (\omega) - 2i\gamma\omega\ \text{x}(\omega) + \omega_0^2\ \text{X}(\omega) = \frac{f(\omega)}{m},$$

Knowing X(ω), we can use the inverse Fourier transform to obtain $x(t)$:

$$x(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-iwt}\, F(w)/m}{-w^2 - 2i\gamma w + w_0^2},$$

Where F(ω) $= \int_{-\infty}^{\infty} dt\ e^{iwt}\ f(t)$

To summarize, the solution procedure for the driven harmonic oscillator equation consists of

(i) Using the Fourier transform on f(t) to obtain f(w) ,

(ii) Using the above equation to find X(w) algebraically ,and

(iii) Performing an inverse Fourier transform to obtain x(t).

To accumulate more intuition about Fourier transform, let us examine the Fourier transform of some interesting function.

Damped Waves:

An exponentially decay function with decay constant $\eta \in R + \eta \in R +$ has the following Fourier transform

$$f(x) = \begin{cases} e^{-\eta x} & ,x\geq 0 \\ 0 & x < 0 \end{cases} \xrightarrow{\text{FT}} \text{f(k)} = \frac{-i}{k - i\eta}$$

Observe that $f(k)f(k)$ is given by a simple algebraic formula. If we "extend" the domain of kk to complex value , $f(k)f(k)$ corresponds to an analytic function with a simple pole in the upper half of the complex plane , at $k = i\eta k = i\eta k = i\eta$

Next , consider a decaying wave with wave –number $q \in R\ q \in R$ and decay constant $\eta \in R + \eta \in R+$.

The Fourier transform is a function with a simple pole at $+i\eta q + i\eta$;

$$f(x) = \begin{cases} e^{i(q+i\eta)x} & ,x\geq 0 \\ 0 & ,\ x < 0 \end{cases} \longrightarrow f(k) = \frac{-i}{k-(q+i\eta)}$$

On the other hand , consider a wave that grows exponentially with xx for x<0 x<0 , and is zero for x>0 x>0. The Fourier transform is a function with a simple pole in the lower half –plane.

$$f(x) = \begin{cases} 0 & ,x \geq o \\ e^{i(q-i\eta)x} & ,x < 0 \end{cases} \xrightarrow{\text{FT}} f(k) = \frac{i}{k-(q-i\eta)}$$

From these examples, oscillations and amplification /decay in $f(x)f(x)$ are related to the existence of poles in the algebraic expression for $f(k)f(k)$.

The real part of pole position gives the wave –number of the oscillation, and the distance from the pole to the real axis gives the amplification or decay constant. A decaying signal produces a pole in the upper half-plane, while a signal that is increasing exponentially with xx produces a pole in the lower half-plane.

In both cases, if we plot the Fourier spectrum of $|F(k)|2|F(k)|2$ versus real kk , the result is a Lorentzian peak centered at k=qk with width 2y2y.

Applications of Fractional Differential Equations:

Applications originate from a diverse array of scientific and technical disciplines.

Fractional differential equations are gaining popularity as a modeling tool for comprehending many elements of nonlocality, including as

(1) Fractional Calculus in Physics:

Fractional derivatives have a role in the mathematical representation of electrical circuits and generalized voltage dividers. Le Mehaute and Crepy proposed that impedance of fractional order.

(2) Fractional Calculus in Electrochemistry and tracer fluid flows:

The Fractional advection-dispersion equation (FADE) is used in the field of groundwater hydrology to simulate the movement of inert tracers transported by the flow of fluid through a porous media.

The dispersion of tracers is heavily influenced by the size of observation. There are three main causes of dispersion: molecular diffusion, fluctuations in the permeability field (micro dispersion), and variations in fluid velocity in a porous media.

Conclusion:

The Caputo fractional derivatives has emerged as one of the most widely used formulations in fractional calculus, primarily because it allows the incorporations of physically meaningful initial conditions expressed in terms of integer-order derivatives. This makes the Caputo definition particularly suitable for modeling real-world processes such as viscoelasticity, anomalous diffusion, and control systems. Compared with the Riemann-Liouville derivative, the Caputo approach provides greater flexibility in physical interpretation and facilitates the applications of standard numerical method to fractional differential equations.

In addition, the Caputo derivatives has been successfully employed in both ordinary and partial fractional differential equations, contributing to the modeling of phenomena where memory effects and hereditary properties play a crucial role. Its compatibility with Laplace transform techniques and its close relation to classical derivatives further enhance its appeal in applied mathematics and engineering contexts.

Future Directions:

Although the Caputo derivative has been extensively studied, several promising directions for future research remain:

Numerical Schemes:

Development of more stable, efficient, and adaptive numerical methods for solving Caputo-type fractional differential equations, especially for nonlinear and high-dimensional systems.

Fractional PDEs:

Extension of Caputo –based methods to multi-dimensional fractional partial differential equations with complex geometries and boundary conditions.

Stochastic Systems:

Investigation of stochastic fractional differential equation involving the Caputo derivative to model random effects and uncertainties in real-world systems.

Hybrid Models:

Exploration of Caputo derivatives in combination with other definition (Riemann-Liouville, Atangana -Baleanu ,Prabhakar derivatives) for hybrid models that capture a wider range of physical behaviors.

Applications in Emerging Fields:

Expanding the use of Caputo derivatives in areas such as machine learning, biomedical sciences, fractional control theory, and financial mathematics, where memory effects are significant.

Analytical Frameworks:

Deeper study of existence, uniqueness, and stability of solutions for Caputo-type fractional differential equations, particularly in nonlinear and variable-order cases.

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References:

1. I. Beltagy, K. Lo, and A. Cohan, "SciBERT: A pretrained language model for scientific text," arXiv Prepr. arXiv1903.10676, 2019
2. A. Giusti et al., "A practical guide to Prabhakar fractional calculus,"Fract. Cale. Appl. Anal., vol.23, no. 1, pp. 9-54, 2020, doi: 10.1515/fca-2020-0002.
3. H. Jahanshahi, A. Yousefpour , J. M. Munoz-Pacheco, I. Moroz , Z. Wei, and O. Castillo, " A new multi-stable fractional- order four- dimensional system with self-excited and hidden chaotic attractors: Dynamic analysis and adaptive synchronization using a novel fuzzy adaptive sliding mode control method," Appl. Soft Comput., vol.87,p.105943,2020.
4. E. IIhan and I. O. Kiymaz," A generalization of truncated M-fractional derivative and applications to fractional differential equations," Appl. Math. Nonlinear Sci., vol. 5, no.1, pp. 171-188,2020
5. V. E. Tarasov, "On History of Mathematical Economics: Application of Fractional Calculus," Mathematics, vol. 7, no. 6. 2019. doi:10.3390/math7060509.
6. L. Lu, X. Meng, Z. Mao and G.E. Karniadakis," DeepXDE: A deep learning library for solving differential equations,"SIAM Rev., vol.63, no. 1, pp. 208-228, 2021.
7. S. Shelare, R. Kumar, T. Gajbhiye, and S. Kanchan, "Role of geothermal energy in sustainable water desalination—a review on current status, parameters, and challenges," Energies, vol. 16, no. 6, p.2901, 2023.