# An Overview of Quantum Mechanics

***Agastya Sumeer Kaul**
****Vikrant Pethia**
*****Neeraj Pant**
******Vijay Kaushik**

## Abstract

This paper looks at the complexity of quantum mechanics and sets out the mathematical and physical basis of the Quantum Harmonic Oscillator (QHO) and its applications. We prove the eigenstate equation inductively, looking at the roles of wave functions and eigenstates alongside operators in quantum mechanics and their importance in Hilbert space. Based on the principle of mathematical induction, by proving the equation $\hbar\omega\left(n+\frac{1}{2}\right)=E$ for any value of the quantum number "$n$" (or "$k$" as per the conventional induction notation, either notation meaning the energy state of a quantum system which is usually a positive integer) and then proving the equation is true for "$k+1$", we establish the recursive/iterative relation between the energy state and the given values and prove the equation is true for all values of "$n$". This proof further confirms that energy states of a quantum harmonic oscillator are discrete and quantised. The main contribution is a clear mathematical link between quantum numbers and energy levels. The quantum harmonic oscillator model based on this axiom is used to approximate many physical systems including molecular vibrations and lattice vibrations in solids. The insight into wave functions allows one to yield the spatial arrangement of their particles, and their probability densities. The outcomes possess very significant implications for theoretical and practical considerations, especially in molecular chemistry and condensed matter physics and open up further avenues for research mainly in complex quantum systems and their behavior. Overall, this paper touches upon the transition from classical mechanics, to its quantum correlative.

**Keywords:** Quantum Harmonic Oscillator, Wavefunction, Eigenstates, Eigenvalues, Operators, Mathematical Induction

## I. Introduction

Visualize your earliest childhood memory. Allowing the wind to gently push the swing upon which you are seated, without a care in the world. Embedded within this lies a vast array of mathematical principles, which are executed in the real world through physics. Within the classical world, one can perceive simple harmonic motion exemplified in this very swing. Making the quantum leap, the familiar becomes unfamiliar. The classical model of predictability, controlled by definable equations, gives way to a probabilistic universe based on wave functions. Moving into that quantum realm, the very basic tenet of quantum mechanics is that eigenstates provide a framework by which one can understand the potential states of a system.

This paper is based upon an inductive proof in relation to the Eigenstates equation. Its axiomatics and applied aspects were analyzed in the investigation, which revealed a corresponding relation between the principles of mathematical induction, and the familiar equations we are accustomed to. Learning about the Eigenstates equation, the concepts of quantum mechanics which allow us to further understand it, and its proof will make us comprehend further about the fundamental nature of quantum systems and how they work.

## II. Simple Harmonic Oscillator

In mechanics, simple harmonic motion[i] is a type of periodic motion or oscillation motion where the restoring force is directly proportional to the displacement and acts in the direction opposite to that of displacement. Let us take a cube with mass "$m$" as a spring mass system. It is placed on a frictionless plane adjacent to a wall, connected by a spring attached to both surfaces. The point at which the aforementioned spring is neither stretched, nor compressed, is known as its equilibrium, "eq".

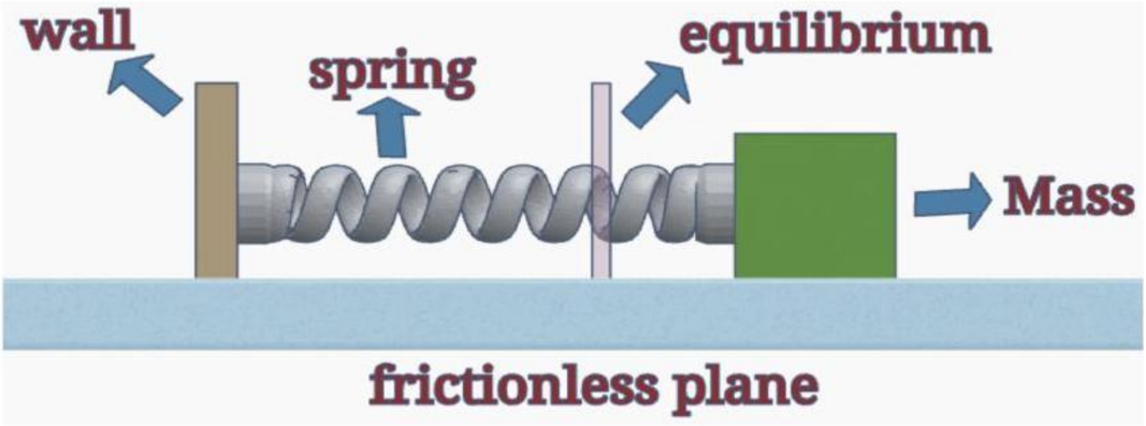


**Figure 1.1- Visualisation of the Simple Harmonic Oscillator**

The distance between the object, and the equilibrium line is "$x$". Since we know forces are a vector quantity, the force applied in the direction facing the wall will be "$-F$". If the mass is displaced from the equilibrium position, the spring exerts a restoring elastic force that obeys Hooke's law.

Hence, recalling Hooke's law, the equation becomes,

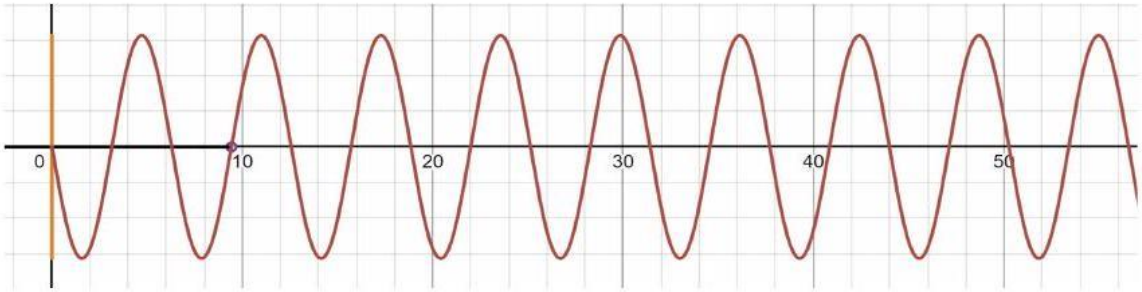
$$F = -kx$$
$$ma = -kx$$
$$a = -\frac{k}{m} \cdot x$$

The formula for the acceleration becomes,

$$\omega = \sqrt{\frac{k}{m}}$$
$$a = -\omega^2 x$$
$$\frac{d^2x}{dt^2} = -\omega^2 x$$

Further, the general equation of this using trigonometric properties where $x_0$ is the amplitude and $\phi$ the phase of the oscillator, with respect to time is,

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$
$$= x_0\cos(\omega t + \phi)$$



When we graph this equation,

**Figure 1.2- Visualisation of the SHO's graphical representation with respect to time, and distance**

In this visualisation, the point following the oscillator's motion reaches its crest when the spring is fully extended, and trough when completely compressed. Before we delve into the quantum aspect of the topic, we must understand the "energy" aspect of the S.H.O.

Recalling some equations,

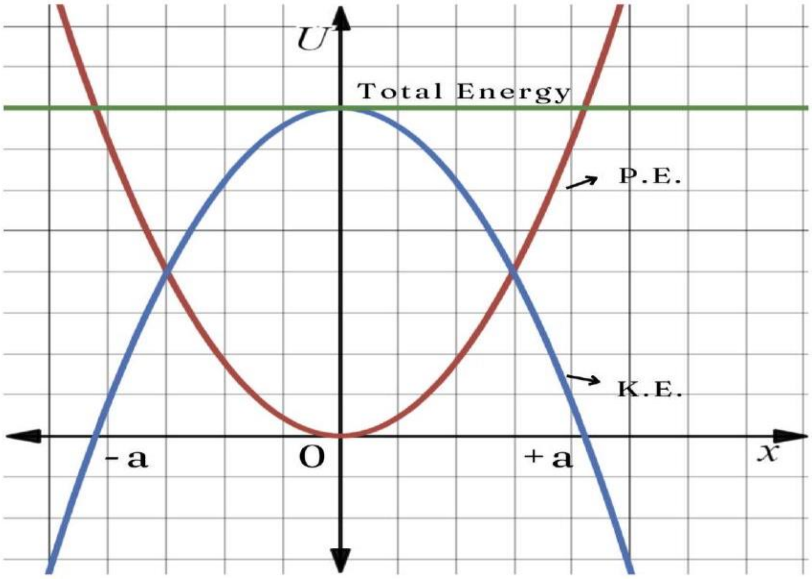
$$K.E. = \frac{1}{2}mv(t)^2$$
$$P.E. = \frac{1}{2}kx(t)^2$$

The total energy of the system results in the equation,

$$E = K + U$$

Now we delve into something quite fascinating. The following proves that the energy in this oscillator is actually a constant[ii] when we take the rate of change of "$E$",

$$\to \frac{d}{dt}\left(\frac{1}{2}mv^2 + \frac{1}{2}kx^2\right)$$
$$\to mva + kxv$$
$$\to v(ma + kx)$$
$$\to v(-kx + kx)$$
$$= 0$$



When we graph this equation,

**Figure 1.3- Visualisation of a Particle with respect to Potential Energy & Extension**

By solving for the time derivative of "$E$" in terms of "$E$" and "$x$",

$$\frac{dx}{dt} = \left(\frac{2E}{m} - \omega^2 x^2\right)^{\frac{1}{2}} = \omega(x_0^2 - x^2)^{\frac{1}{2}}$$

The visualisation, paired with the equation above displays how a particle, which is bound on the parabolic curve, will have a positive acceleration until it reaches the point "$E$" (the horizontal green line) on the $y$-axis, at which it will then accelerate negatively as the motion of the object will be in the opposite direction. In other words, the particle will start from rest at turning point ( $x = \pm x_0$ ), pick up speed until it reaches the origin, and slow down to rest by the time it reaches the other turning point. The simple harmonic oscillator displays that any object near equilibrium will sinusoidally oscillate in the motion we observed until it reaches that point. In the example we analysed, there was a frictionless plane, and we took the assumption that no energy was lost or gained by the system, thus the harmonic motion would continue until time immemorium.

## III. Wavefunctions

In quantum mechanics, wavefunctions[iii] are fundamental mathematical descriptions that encapsulate the quantum state of a physical system. The wave function provides complete information about the system's properties, particularly the probabilities of finding particles in various positions, momenta, and other physical quantities. By applying operators (e.g., position, momentum, energy) to the wave function, expected values of physical quantities can be derived. Upon measurement, the wave function "collapses" to a specific state corresponding to the observed value, after which the system is described by a new wave function. Regarding our quantum system, denoted as "$\Psi$", its corresponding state vector is denoted as "$|\Psi\rangle$", with "$|\mathrm{I(t)}\rangle$" being the state vector at a particular point in time.

In one dimensional space, wave functions are typically denoted by $\Psi(x,t)$, "$x$" being the position of the system and "$t$" referring to time. In a 3 dimensional space, the same representation takes the form of $\psi(r,t)$, wherein "$r$" is expanded to $(x,y,z)$, i.e. the position vector in 3D space. A wave function must be normalised so that the total probability of finding the particle somewhere in space is equal to 1. For a one-dimensional system, this is expressed as:

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1$$

In three dimensions, it becomes,

$$\int_{-\infty}^{\infty} |\psi(r,t)|^2 d^3r = 1$$

Under the principles of superposition, a quantum system can exist in a linear combination of multiple states simultaneously, allowing wave functions to be added together to form new wave

functions. In the superposition of quantum states, the relative phase between the states determines the resulting quantum state. For instance, two states $\Psi_1(x)$ and $\Psi_2(x)$ can combine to form a new state $\psi(x) = \psi_1(x) + e^{i\phi}\psi_2(x)$, where "$\phi$" is the relative phase between the two states. The principle of orthogonality states that if a system is in the state $\psi_n(x)$ it cannot simultaneously be in the state $\Psi_m(x)$, provided that $\Psi_n(x)$ and $\Psi_m(x)$ are orthogonal, i.e., their inner product is 0.

$$\langle \psi \mid \phi \rangle = 0 \text{ if } |\psi\rangle \text{ and } |\phi\rangle \text{ are orthogonal}$$

This orthogonality condition ensures that these eigenstates are distinct and non-overlapping in the Hilbert space, where quantum states are mathematically represented. For continuous systems, like a particle in a box, the orthogonality condition applies to wave functions $\Psi_n(x)$ and $\Psi_m(x)$ associated with different quantum numbers[iv] "$n$" and "$m$":

$$\int_{-\infty}^{\infty} \Psi_n^*(x)\Psi_m(x)dx = 0 \text{ for } n \neq m$$

Where, $\Psi_n^*(x)$ is the complex conjugate of the wavefunction $\Psi_n(x)$, and the integral is taken over the entire space. This integral being zero signifies that the wavefunctions $\Psi_n(x)$ and $\Psi_m(x)$ are orthogonal. The ket representing a physical state can be expanded in terms of $|x'\rangle$, $|\alpha\rangle = \int dx' |x'\rangle\langle x' \mid \alpha\rangle$ and $|\langle x' \mid \alpha\rangle|^2 dx'$ is the probability of the particle being found in a narrow interval "$dx'$" around "$x$", and in this scenario the inner product "$\langle x' \mid \alpha\rangle$" is what is referred to as the wave function $\psi_\alpha(x')$ for the state $|\alpha\rangle$: $\langle x' \mid \alpha\rangle = \Psi_\alpha(x')$. A wave function, typically a complex function also consists of an imaginary part and is wholly represented as:

$$\psi(x,t) = R(x,t)e^{i\theta(x,t)}$$

$R(x,t)$ being the amplitude and $\theta(x,t)$ being the phase. The phase of a wave function is crucial in interference patterns, where the superposition of two or more wave functions leads to constructive or destructive interference. Constructive interference occurs when the phase difference between wave functions is an integer multiple of $2\pi$, leading to an increase in amplitude. Destructive interference occurs when the phase difference is an odd multiple of $\pi$, leading to a decrease or cancellation of amplitude.

## IV. Hilbert Space

Revisiting the swingset analogy, we may observe the environment around us as a container, one that defines our characteristics, position and so on. This simple analogy is reflected in the quantum realm in the form of the Hilbert space, an abstract vector space which is not bound to the same laws as

Euclidean spaces as it can have an infinite number of dimensions, which allow them to accommodate complex quantum states.

In this unbounded space, the wave function or state vector of the system lives wherein all the possible states of a system are represented. The inner product of Hilbert space allows us to calculate probabilities and expectation values. So, when we measure a quantum system, we get that particular state, and it has the probability which is proportional to the square of the inner product between that system and the present state, by which the given state has been measured. Hence, a state vector inside Hilbert space is all equivalent to a wave function: squaring such, we can find out the graph of probabilities about that particle. The position, momentum, and energy observables are represented by operators acting on vectors in Hilbert space. If the operator acts on an eigenvector, then the operator returns a scalar multiple, the eigenvalue of the vector, which represents the measured value of that observable.

The eigenvalue is the factor by which the span of an eigenvector is scaled up or down when acted upon by an operator. The Hilbert space is said to be 'complete' in the sense that it encompasses any and all possible states of a system as a combination of base vectors, in accordance with the principles of superposition.

Mathematical environs that rigorously define a quantum mechanical state, operators, and all physical observables are of high importance in Hilbert space in proving the equation conveying an eigenstate for such systems as the Quantum Harmonic Oscillator. Upon the development of such basic relationships as quantized energies, one can prove relations connecting such a system with others.

An inner product is denoted as "$(X, \langle\cdot,\cdot\rangle)$", in which "$X$" is a vector space, and "$\langle\cdot,\cdot\rangle$" is an inner product. Firstly, the properties of inner products will be analysed. $F \in \{R, C\}$, let $X$ be an $F -$ vector space.

$$1.\quad \langle x, x\rangle \geq 0 \text{ for all } x \in X \text{ and } \langle x, x\rangle = 0 \Leftrightarrow x = 0$$

The first property of inner products is Positive Definiteness. If both sides are taken as the same vector "$x$", in real and complex vector spaces, the inner product is a non-negative, real number; implying that the norm of any vector, resulting from the inner product, will be non-negative. The only vector for which this norm becomes zero is the zero vector itself.

$$2. \begin{array}{l} \langle x, y\rangle = \langle y, x\rangle \text{ for all } F = R \\ \langle x, y\rangle = \overline{\langle y, x\rangle} \text{ for all } F = C \end{array}$$

The second property of inner products is Conjugate Symmetry. In the case of inner products on real vector spaces, they are symmetric. Thus, the order in which the vectors are taken does not make any difference to the result. For the inner product spaces on complex vector spaces, however, there will be conjugate symmetry. If the vectors are interchanged, the inner product changes to the complex conjugate; ensuring the result is a non-negative real number. This property generalizes the geometric meaning of the inner product in complex spaces to give a value that is consistent with the real-valued length and angle measurements.

$$3. \langle x, y_1 + y_2\rangle = \langle x, y_1\rangle + \langle x, y_2\rangle \text{ for all } x, y_1, y_2 \in X$$

$$\langle x, \vartheta y\rangle = \vartheta\langle x, y\rangle \text{ for all } \vartheta \in F \;\&\; x, y \in X$$

The third property of inner products is linearity in the 2nd argument, and the scalar must be homogenous. The inner product is linear with respect to the second argument; it obeys additivity and homogeneity properties. For a scalar, the inner product is homogeneous, and the scalar may be factored out. This provides predictability to an inner product under linear transformations.

Now that inner products have been established, norms can be introduced. Norms are a way to measure the length of a vector. Firstly, the properties of this will be analysed. $F \in \{R, C\}$, let $X$ be an $F -$ vector space.

A map $\| \cdot \|: X \to [0, \infty)$ is called a norm if -

The properties of Positive Definiteness and Absolute Homogeneity ensure that any norm is non-negative, and scales homogeneously under scalar multiplication. These were defined previously.

1. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$

The triangle inequality[v] says that the norm of a sum of two vectors can never be larger than the sum of their norms. This provides the classical triangle inequality in Euclidean space; the length of one side of a triangle is always less than or equal to the sum of the lengths of the other two sides. To simplify the aforementioned, we can visualise 2 arrow vectors forming a right-angle triangle. The distance, according to classical mathematics, must be the sum of the magnitude of both these vectors (modulus as they cannot be negative or complex numbers).

If $(X, d_{\|\cdot\|})$ is a complete metric space, then the normed space $(X, \| \cdot \|)$ is called a Banach space. Another relatively interesting correlation is that this space is also a complete metric space. A cauchy sequence is a sequence which is converging, meaning it converges/intersects with a point in the vector space while the series moves closer to zero (however, not every cauchy space is a converging sequence). The norm of "$d_{\|\cdot\|}$" creates this exact series, meaning there are no holes in the series, and this gives the normed space we defined earlier to now be called a Banach Space.

Essentially, a Hilbert Space is a generalisation of the norm and length of vector spaces from 2 dimensions, to an infinite number of dimensions.

$(X, \langle \cdot, \cdot \rangle)$ is called a Hilbert space if $(X, \| \cdot \|_{\langle \cdot, \cdot \rangle})$ is a Banach space.

A Hilbert space is a Banach space with a norm derived from an inner product. Provided a Banach space is there with a norm coming from an inner product and complete with respect to this norm, it is a Hilbert space. It means that the space should satisfy all axioms of a Banach space-that is, completeness with respect to the norm-and at the same time have an inner product structure generating this norm. Hilbert space is an n-dimensional vector space spanned by wave (eigen) functions $\}\{\Psi_j\}$. The Hilbert space is the mathematical structure within which the quantum states of the QHO are represented. It is a complete, infinite-dimensional vector space equipped with an inner product. This setting will mathematically define the quantum states to be vectors (or wavefunctions) and the observables to be operators acting on these vectors rigorously. Furthermore, the superposition principle is related in an essential way with the Hilbert space structure. Any quantum state of the QHO can be expressed as a linear combination of the basis states in Hilbert space. The above principle gives a description of complex quantum phenomena, including interference and entanglement.

## V. Copenhagen Interpretation

The Copenhagen Interpretation[vi] is essentially a collection of views regarding quantum mechanics, stemming from the work of Niels Bohr, Werner Heisenberg, Max Born, and others. The concerned sections are the "Observer's Effect", and "Heisenberg's Uncertainty". For instance, let us take a simple radio. It comprises a knob that allows one to change the frequency which is transmitted and received, enabling us to change the channel. Until we turn the knob, the radio is locked in a state in which it can potentially emit any frequency of radio waves, however, it is ambiguous to us until we use it. In terms of the QHO, the position of a quantum system at a particular period of time is known to be "superpositioned". This means that until a measurement is carried out, the particle is in all the potential states at the same time. The most notable example of this is Schrödinger's Cat. The application of the observer's effect is that while an entity is either "measuring" or "observing" the object, it is disturbing/altering the natural behaviour of the quantum system. When a quantum particle is measured, it results in the wavefunction to forcibly collapse and choose a definite state (like a specific position or momentum), thus the probability of finding the particle anywhere else drops to zero. Since the probability distribution has changed, the

wavefunction fundamentally changes in the same way. Unlike classical physics, making measurements in quantum mechanics has definite consequences. However, quite abruptly, the probability spreads across various positions, thus the whereabouts of the particle becomes more ambiguous as time passes. This change is in accordance with the Schrödinger Equation, which describes the wave function's behaviour with respect to time.

## VI. The Schrödinger Equation

In mathematical physics, the Schrödinger equation represents the basic equation of nonrelativistic quantum mechanics, similar to the function of Hamilton's laws of motion in non-relativistic classical mechanics. The equations of quantum field theory are in a certain sense the relativistic quantum mechanics equivalent to Heisenberg's equation; that is, Schrödinger's equation does not have a natural, direct analogue. In pure mathematics, the Schrödinger equation and its variants make up one of the basic equations studied in the subject area of partial differential equations, and applications made to geometry, spectral and scattering theory, and integrable systems. There exist two, closely related variants of Schrödinger's equation; the time-dependent Schrödinger equation and the time-independent Schrödinger equation. The Time-Independent Schrödinger Equation is defined below as[vii]:

$$\frac{-\hbar^2}{2m}\frac{d^2\Psi}{dx^2}+V\Psi=E\Psi \Rightarrow -\frac{\hbar^2}{2m}\frac{d^2\Psi(x)}{dx^2}+\frac{1}{2}m\omega^2x^2=E\Psi$$

This is the equation of a quantum system in a stationary state. The wave function "$\Psi(x)$" is only dependent on the position "$x$", and not time. Solutions, represented by "$\Psi(x)$", are associated with the spatial distributions of a given particle, while "$E$" provides the eigenvalue of energy for that state. The Time-Dependent variant of this is essentially a linear combination of the

aforementioned eigenstates (stationary states), however with the introduction of time-dependent coefficients. For instance, in videography we create the illusion of an object moving through a series of still images, which are strung together to create motion. Similarly, we can predict the way a wavefunction fundamentally changes with respect to time by stringing together a series of the still eigenstates.

Firstly, the total wavefunction can be separated into its two components; the spatial and time-dependent aspect:

$$\Psi(x,t)=\psi(x)\cdot\phi(t)$$

Now when we substitute the separated form into the time-dependent Schrödinger equation:

$$i\hbar\frac{d}{dt}[\psi(x)\phi(t)] = \hat{H}[\psi(x)\phi(t)]$$

Since $\psi(x)$ is time-independent, turning it into a time-dependent variable:

$$\psi(x)i\hbar\frac{d\phi(t)}{dt} = \phi(t)\hat{H}\psi(x)$$

Simplify and rearrange to:

$$i\hbar\frac{d\phi(t)}{dt} = E\phi(t)$$
$$\hat{H}\psi(x) = E\psi(x)$$

The time-dependent part of the equation can be further solved as:

$$\psi(t) = e^{\frac{-iEt}{\hbar}}$$
$$\Psi(x,t) = \psi(x)\cdot e^{\frac{-iEt}{\hbar}}$$

The solution "$\hat{H}\psi(x) = E\psi(x)$" of the equation is sometimes viewed as stating that if one operates with the Hamiltonian operator "$\hat{H}$" on the wave function "$\Psi(x)$", one obtains the same wave function multiplied by a scalar factor "$E$". Hence, "$\psi(x)$" is an eigenfunction of the Hamiltonian, with "$E$" the associated eigenvalue, representing just one such energy level in the system. The factor "$e^{\frac{-iEt}{\hbar}}$"introduces a time-dependent phase into the wave-function, which will be the exponential time-dependent factor which determines the evolution of our quantum state, provided it has definite energy.

The equation below is a slightly more complicated variant of the Hamiltonian operator, which includes the relativistic term, spin orbit coupling, and Darwin term. However, these corrections typically are small for ordinary QHOs except in the case of a very high frequency oscillator (high $\omega$) or in the case when the mass of the particle is great enough for it to have significant relativistic effects:

$$-\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega^2 r^2 + \frac{-\hat{p}^4}{8m^3c^2} + \frac{1}{2m^2c^2r}\frac{1}{r}\frac{dV}{dr}\cdot\hat{S}\cdot\hat{L} + \frac{\hbar^2}{8m^2c^2}\nabla^2 v = \hat{H}$$

Thus, these extra terms are usually nullified as they are quite insignificant to the final energy values. Due to this condition, the final time-dependent variant of the Schrödinger equation comes out to be:

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(x,t)\right]|\Psi(x,t)\rangle = i\hbar\frac{\partial}{\partial t}|\Psi(x,t)\rangle$$

## VII. The Quantum Harmonic Oscillator

Now that we have established the basics, we can move onto the quantum harmonic oscillator. The quantum harmonic oscillator (QHO) is an essential physical modelling tool for a wide range of theoretical and experimental problems from electrodynamics to cosmology. It is essentially the quantization of the simple harmonic oscillator. The QHO is one of the most significant model systems in quantum mechanics because a given smooth potential can generally be represented as a harmonic potential at a vicinity of a stable equilibrium point. The tiny vibrations in a diatomic molecule are classically viewed using a quantum harmonic oscillator model, but the representation is universal in the sense that it may be extended to a range of other circumstances in physics and beyond. For instance, let us picture a single row of stationary cations (positively charged ions) in a metal, engulfed in a sea of delocalised electrons. One will notice that this creates potential energy wells between the particles, which is the premise of our QHO. This translates into these potential wells further being approximated as Quantum Harmonic Oscillators, which can enable us to predict the possible locations of the said electron.

Provided with the dynamics by the potential energy, the square is in proportion to its displacement from the equilibrium position. As the QHO has exact solutions for its energy levels and wave functions, it proves important as a basis for inquiry into vibrational modes of molecules and phonons in solids, if not for approximating a great variety of other quantum systems. It then defines the potential behaviour of a particle to be bound by a parabolic potential well and goes on to guarantee its energy states are quantized. Quantization essentially means a particle is capable of having an energy level that is discrete; however, not as in classical mechanics, arbitrary energies.

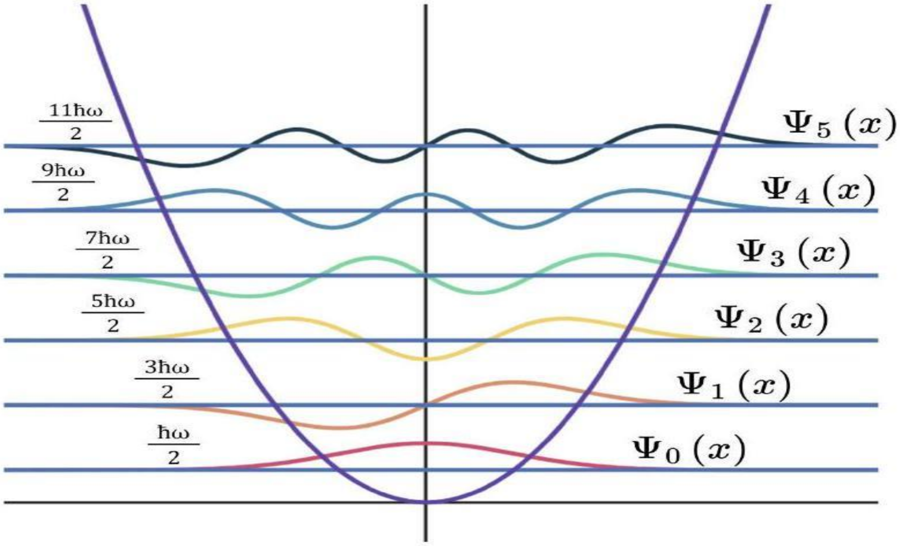


**Figure 1.4- Visualisation of the Potential Energy Well of a QHO**

In **Figure 1.4**, the potential energy well of the quantum harmonic oscillator with quantized energy levels and wave functions "$\Psi_n(x)$" which describe the probability distributions associated with these quantum states are displayed. The parabolic curve depicts the potential energy $V(x) = \frac{1}{2}m\omega^2x^2$, "$m$" is the mass of the particle and "$\omega$" angular frequency of the oscillator. In the potential well, the crossing horizontal lines indicate discrete energy levels of the system, given by the expression[viii] "$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$" and these are parted by intervals of fixed "$\hbar\omega$", and hence shows that in the QHO nature energy is quantized; meaning one can have certain fixed amounts of energy are allowed. The wave functions describe the probability amplitudes[ix] for the particle to be found at different positions within the well potential. In the ground state, "$\Psi_0$" corresponding to the lowest possible energy, the wave function is centered around the minimum potential well and relatively narrow, showing a higher probability of finding the particle near the center. The even-odd alternate forms indicate symmetry with the parabolic potential in every higher energy level-the so-called alternate even and odd forms. Even-numbered states $(\Psi_0, \Psi_2, \Psi_4)$ are symmetric about the center; odd-numbered states $(\Psi_1, \Psi_3, \Psi_5)$ are antisymmetric. Once again, that the lowest possible energy is not zero but "$\frac{1}{2}\hbar\omega$" shows the existence of zero-point energy, which means one can't keep the particle at rest even in a ground state due to basic uncertainties imposed by the Heisenberg uncertainty principle.

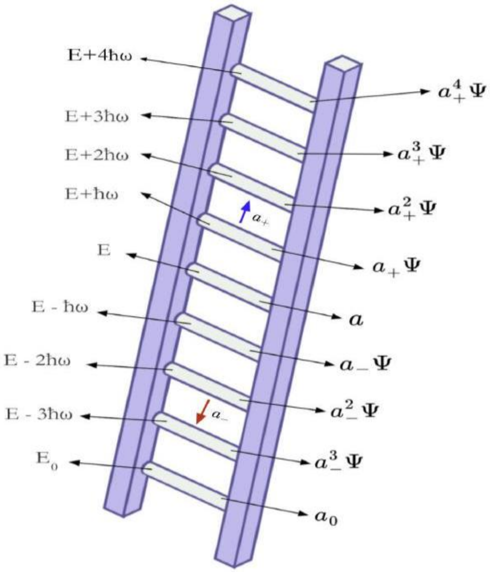


**Figure 1.5- Ladder of Stationary States in the Harmonic Oscillator**

*Creation Operator (* $a_+$*or* $a^\dagger$ *):* $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$

*Annihilation Operator (*$a\_$ *or* $a$*):* $a|n\rangle = \sqrt{n}|n-1\rangle$

The operators " $a$ " and "$a^\dagger$" represent ladder operators and have the effect of lowering or raising the energy of the state. In fact, the operator representation achieves something quite remarkable and, as we will see, unexpectedly profound. The quantum harmonic oscillator describes the motion of a single particle in a one-dimensional potential well. Its eigenvalues turn out to be equally spaced; a ladder of eigenvalues, separated by a constant energy "$\hbar\omega$". If we are energetic, we can of course translate our results into a coordinate representation "$\Psi_n(x) = \langle x \mid n\rangle$[7]". However, the operator representation affords a second interpretation, one that lends itself to further generalization in quantum field theory. We can instead interpret the quantum harmonic oscillator as a simple system involving many fictitious particles, each of energy "$\hbar\omega$". In this representation, known as the Fock space, the vacuum state "$|0\rangle$" is one involving no particles, "$|1\rangle$" involves a single particle, "$|2\rangle$" has two and so on. These fictitious particles are created and annihilated by the action of the raising and lowering operators, "$a$" and a with canonical commutation relations, "$[a, a^\dagger] = 1$". In **Figure 1.5**, we observe the stationary states/energy levels of the quantum harmonic oscillator visualised as a ladder to aid in one's visualisation in the ladder operator's effects on the eigenstates of the particle. On the left side of the ladder, these eigenstates are represented in terms of its eigenvalues, while on the right side it is represented as the actions of the ladder operators. "  " and " $a$ " is the current energy value of the particle, and, " $E_0$ " and " $a_0$ " is the ground level of the quantum system. To move up an energy level, we apply a creation operator to our wavefunction, denoted as " $a_+\Psi$ ", and so on. To move down an energy level, we apply an annihilation operator to our wavefunction, denoted as " $a\_\Psi$ ", and so on. The aforementioned can be expressed as energy values, as we know the difference between energy levels are " $\hbar\omega$", we can add or subtract this expression's multiples depending on the level we want to ascend or descend to.

### VIII. Eigenstates, Eigenvectors, & Eigenvalues

Within the scope of linear algebra, an eigenvector is any vector in space which, when subjected to a specific transformation, experiences alteration only in terms of its length or scale. After the transformation, the eigenvector remains aligned with its original span in the untransformed space. The factor by which the length of an eigenvector increases or decreases in terms of length after transformation is known as the eigenvalue for that specific eigenvector under the given transformation.

Eigenstates, under the purview of quantum mechanics, refer to certain states of a quantum system such that when the state is acted upon by an operator (such as the momentum or energy operator), the result acquired is a multiplication of the original state by a constant value, which is the eigenvalue in this scenario. A case wherein this concept is observed is that of the quantum harmonic oscillator. A quantum harmonic oscillator can be viewed as a particle inside a probability well, i.e., a graph which increases quadratically with increase in displacement from origin. Unlike the particle in the well which may move freely along the graph curve, the quantum system has an energy level that is quantised and can only be equal to certain values. The energy level of the system here is its eigenstate and the wavefunction of the system is considered an eigenvector within the Hilbert space.

The eigenstates of the quantum harmonic oscillator are denoted by $\Psi_n(x)$, where $n$ is a non-negative integer $(0,1,2,\dots)$. Each $\Psi_n(x)$ is associated with a different energy level and has a specific form, typically involving Hermite polynomials[x] and a Gaussian function. The eigenvalues associated with these eigenstates are given by:

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

***The derivation of this equation is shown below.***

The time-independent Schrödinger equation for a particle in a potential $V(x) = \frac{1}{2}\omega^2 x^2$ is

$$\hat{H}\Psi(x) = E\Psi(x)$$

Where the Hamiltonian operator $\hat{H}$ is: $\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$

In terms of the position representation, the momentum operator: $\hat{p} = -i\hbar\frac{\partial}{\partial x}$

Therefore, the Hamiltonian becomes: $\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2$

Thus, the Schrödinger equation is: $-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x)}{\partial x^2} + \frac{1}{2}m\omega^2 x^2 = E\Psi(x)$

To simplify the problem, we introduce a dimensionless variable: $\xi = \sqrt{\frac{m\omega}{\hbar}}x$

This substitution scales the position " " by a factor that involves the mass " $m$ ", the angular frequency " $\omega$ ", and the reduced Planck constant " $\hbar$ ".

The partial derivative with respect to " $x$ " transforms as follows:

$$\frac{\partial}{\partial x} = \sqrt{\frac{m\omega}{\hbar}}\frac{\partial}{\partial \xi}$$

Thus, the second derivative is:

$$\frac{\partial^2}{\partial^2 x} = \left(\sqrt{\frac{m\omega}{\hbar}}\right)^2 \frac{\partial^2}{\partial \xi^2}$$

Substitute $\frac{\partial^2}{\partial x^2}$ into the Schrödinger equation:

$$\frac{-\hbar^2}{2m} \cdot \frac{m\omega}{\hbar} \frac{\partial^2 \psi(\xi)}{\partial \xi^2} + \frac{1}{2} m\omega^2 \left(\frac{\hbar}{m\omega} \xi^2\right) \Psi(\xi) = E\Psi(\xi)$$

Simplify and rearrange to:

$$-\frac{\partial^2 \psi(\xi)}{\partial \xi^2} + \xi^2 \Psi(\xi) = \frac{2E}{\hbar\omega} \Psi(\xi)$$

$$\text{Let: } \frac{2E}{\hbar\omega} = \lambda_n$$

Thus, the equation takes the form of the Hermite Differential Equation:

$$-\frac{\partial^2 \psi(\xi)}{\partial \xi^2} + \xi^2 \Psi(\xi) = \lambda \Psi(\xi)$$

To solve this equation, we look for solutions of the form:

$$\Psi_n(\xi) = N_n H_n(\xi) e^{\frac{-\xi^2}{2}}$$

where " $H_n(\xi)$ " are Hermite polynomials, and " $N_n$ " is a normalisation constant.

Hermite polynomials " $H_n(\xi)$ " are solutions to the Hermite differential equation where " $n$ " is a non-negative integer:

$$\frac{d^2 H_n(\xi)}{d\xi^2} - 2\xi \frac{dH_n(\xi)}{d\xi} + 2nH_n(\xi) = 0$$

From the differential equation, the eigenvalues "  " correspond to the quantized values that ensure the polynomial solutions " $H_n(\xi)$ " are valid. By solving this equation, we find:

$$\lambda_n = 2n + 1$$

Substitute " $\lambda_n$ " back into the expression for " $E$ ":

$$\frac{2E_n}{\hbar\omega} = 2n + 1$$

$$\Rightarrow E_n = \left(n + \frac{1}{2}\right) \hbar\omega$$

Wherein " $E_n$ " is the energy eigenvalue corresponding to the "$n^{th}$" eigenstate " $\Psi_n(x)$ ", " $\omega$ " is the angular frequency of the oscillator and " $n$ " is the quantum number ( 0,1,2, ... ), representing different energy levels. These eigenvalues indicate that the energy levels of the quantum harmonic oscillators are quantized, meaning the system can only have specific, discrete energy values.

Since we have established the derivation of the primary equation this paper revolves around, we must indulge in the parameters which ensure that it behaves in the Hilbert Space:

$$\langle \lambda | n | \lambda \rangle = \lambda \langle \lambda \mid \lambda \rangle = \lambda$$

The equation above displays that "**λ**" essentially represented the eigenvalue of " $n$ ", or in other words it gives us the expected value of "$n$", during the eigenstate " $|\lambda\rangle$ ". We can display the positive definiteness of the eigenvalue "**λ**" as:

$$\langle\lambda|n|\lambda\rangle = \lambda\langle\lambda|a^{\dagger}a|\lambda\rangle = \|\hat{a} \mid \lambda\|^{2} \geq 0 \ \Rightarrow \text{Hence}, \lambda \geq 0$$

The zero condition of the eigenstate is described as follows:

$$\hat{a}|\lambda\rangle = 0 \ \text{ if } \lambda = 0$$

In this case the state applied to by the annihilation operator gives the null vector, so the result says that in case " $= 0$ ", " $|\lambda\rangle$ " would correspond to the system's ground state. Finally, the quantization of the eigenvalue "**λ**" is described as:

$$\hat{a}|\lambda\rangle = \sqrt{\lambda}|\lambda - 1\rangle \Rightarrow \lambda = \lambda_{\min} + n \Rightarrow \lambda \text{ integer}$$

It means that $\lambda$ is quantized, and it can only take discrete integer values starting with $\lambda_{\min}$ , as required in the quantum harmonic oscillators.

## IX. Hermite Polynomials

In physics, Hermite Polynomials are a classical orthogonal polynomial sequence that arise in probability, where they give rise to the eigenstates of the quantum harmonic oscillator. The "physicist's" Hermite Polynomials are given by:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

This representation displays that the polynomials are the "n<sup>th</sup>" derivative of the Gaussian function "$e^{-x^2}$" multiplied by the exponential factor "$e^{x^2}$", and a sign alternating factor " $(-1)^n$ ". On the other hand, in this form, one can see very clearly that Hermite polynomials occur at most naturally in contexts where differential operators and Gaussian functions interplay, particularly the solutions of the Schrödinger equation for the quantum harmonic oscillator.

$$H_n(x) = \left(2x - \frac{d}{dx}\right)^n \cdot 1$$

This equation states that Hermite polynomials are derived by iteratively applying the operator "$\left(2x - \frac{d}{dx}\right)$", "$n$" times to the constant function 1. There is some notation in the above expressions where "$x$" is the coordinate position and "$\frac{d}{dx}$" is the differentiation operator with respect to "$x$". The "$\left(2x - \frac{d}{dx}\right)$" operator is called a ladder operator in the QHO as it raises or lowers the quantum states.

In operator form, it encapsulates the recurrence relations which are inherent to the polynomials. Each polynomial in the sequence can be derived from the previous one through the application of this specific operator. Endowed by ladder operators "$a^{\dagger}$" and "$a$", they are used to describe the sequence of creation or annihilation of quanta in the system, as stated earlier. The final form of Hermite polynomial in the operator form reflects the same idea, a link from algebraic structure of energy eigenstates of QHO to mathematical properties of Hermite polynomials.

By tradition, the arbitrary multiplicative factor is chosen so that the coefficient of the highest power of $r$ is $2^n$. The generative function, which allows us to derive the equations of each of the

Hermite Polynomials is;

$$e^{-z^2+2zx} = \sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(x)$$

In the equation above, "$z$" is a dummy variable which is used to generate the Taylor series of the exponential on the left. Since the "$k^{th}$" derivative with respect to "$z$" of the series eliminates all powers with $n < k$, retains $z^{n-k}$ terms for $\mathrm{n} > \mathrm{k}$ and reduces the term in $z^k$ to $\frac{k!}{k!} H_k(x) = H_k(x)$, if we take the "$k^{th}$" derivative and then set $z = 0$, only $H_k(x)$ remains.

The first four Hermitian polynomials are given below with their derivations, given that $z = 0$,

$H_0(x) = e^0$

$= 1$

$H_1(x) = \frac{d}{dz} e^{-z^2+2zx}$

$= (-2z + 2x)e^{-z^2+2zx}$

$= 2x$

$H_2(x) = \frac{d^2}{d^2z} e^{-z^2+2zx}$

$= [(-2 + (-2z + 2x)^2]e^{-z^2+2zx}$

$= 4x^2 - 2$

$H_3(x) = \frac{d^3}{d^3z} e^{-z^2+2zx}$

$= [(8z - 8x) + (-2z + 2x)(4z^2 - 8zx + 4x^2 - 2)]e^{-z^2+2zx}$

$= 8x^3 - 12x$

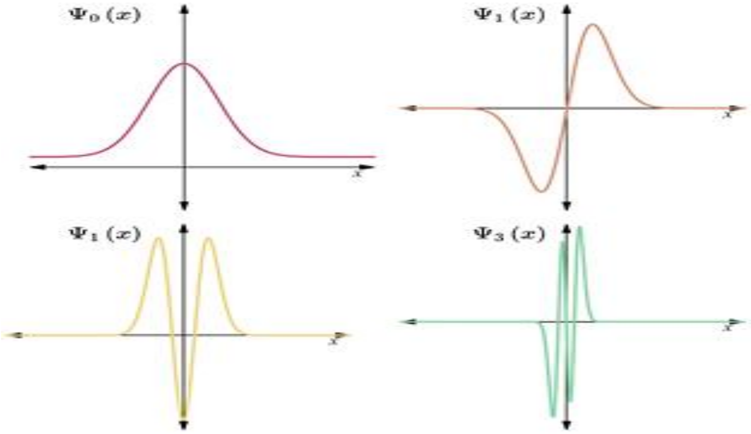


**Figure 1.6- Wave functions associated with the first four levels of the harmonic oscillator**

In **Figure 1.6**, we see the quantum wave functions of a harmonic oscillator at different quantum states. The wave functions describe the probability distribution of the particle position at different energy levels of a harmonic oscillator potential. The ground state converges to a smooth bell-shaped profile indicating that, at "$x = 0$", the profile is at the highest probability density point. The higher the quantum number ($\Psi_1, \Psi_2, \Psi_3$), the more complex the wavefunctions, more oscillations and nodes (points where the wave function passes through zero) are seen. These nodes are places where the probability of finding the particle is zero. High energy states show more pronounced oscillatory behavior and more nodes, so clearly particles in high energy states are more mobile, occupying more positions where they might be found. We see from this figure that when "$n$" increases, the region of the "$Ox$" axis in which "$\Psi_n(x)$" takes on non-negligible values becomes larger. This corresponds to the fact that in classical mechanics, the amplitude of the particle's motion increases with the energy. It follows that the mean value of the potential energy grows with , since "$\Psi_n(x)$", when "$n$" is large, takes on non-negligible values in regions of the $z$-axis where "$V(x)$" is large. Moreover, we see in these figures that the number of zeros of "$\Psi_n(x)$" is "$n$".

The quantum oscillator is strikingly different from its classical counterpart- not only are the energies quantized, but the position distributions have some bizarre features. For instance, the probability of finding the particle outside the classically allowed range (that is, with x greater than the classical amplitude for the energy in question) is not zero, and in all odd states the probability of finding the particle in the centre of the potential well is zero. Only at relatively large "$n$" do we begin to see some resemblance to the classical case.

Relating the QHO to functional analysis, the Hermite Polynomials can be proved to be part of a Hilbert Space via the following-

The norm of the function may be written as;

$$\int_{-\infty}^{\infty} |f(x)|^2 \omega(x) dx < \infty$$

Simply states that the integral of the square of the absolute value of a function $f(x)$, weighted by $\omega(x)$, over the entire real axis, is finite. In quantum mechanics or functional analysis, this condition guarantees that $f(x)$ is a square-integrable function; that is, it belongs to the Hilbert space $L^2(R, \omega(x)dx)$. Essentially, this describes how it has a finite norm and may be considered a "well-behaved" function in this space.

This function's inner product definition may be written as:

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\omega(x) dx$$

The equation defines an inner product of two functions, $f(x)$ and $g(x)$, within a weighted function space. The inner product generalizes the concept of dot product for vectors onto functions and roughly speaking involves integration of the product of $f(x)$ with the complex conjugate of $g(x)$ weighted by $\omega(x)$. This inner product structure underlies many of the geometric and algebraic properties of Hilbert spaces and introduces meanings of orthogonality between functions.

Finally, the orthogonality condition may be described as:

$$\int_{-\infty}^{\infty} f(x) x^n e^{-x^2} dx = 0$$

for every $n \geq 0$, then $f = 0$

The equation simply states that the integral of $f(x)$ with "$x^n e^{-x^2}$" is zero for every non-negative integer "$n$". This implies that $f(x)$ is orthogonal to all functions of the form " $x^n e^{-x^2}$ " over the entire real line. This, in quantum mechanics, may correspond to the orthogonality of wave functions or basis functions in a function space. The conclusion "if this is true for every $n \geq 0$, then $f(x)$ must be zero" assumes that the set of functions "$xe^n e^{-x^2}$", which are related to Hermite polynomials, is a complete orthogonal set in the space of square-integrable functions. The function $f(x)$ has to be the zero function if the system is orthogonal to the complete set. This is essential as the orthogonal property of completeness proves that the system is set in a Hilbert vector space.

With this convention, the normalized stationary states for the harmonic oscillator are:

$$\phi(x) = \frac{A}{\sqrt{2^n n!}} \left(\frac{K}{\pi}\right)^{\frac{1}{4}} (H_n) e^{\frac{-r^2}{2}} + \hbar\omega \left(n + \frac{1}{2}\right)$$

"$A$" is a normalisation constant for the wavefunction. This ensures that the probability of finding a particle anywhere in space and time must be equal to 1. The constant "$\sqrt{2^n n\,!}$" normalisation constant for the Hermitian polynomial, ensuring the integral of the square over all space is also equal to 1. The term " $H_n$ " is the Hermitian Polynomial for the specific energy level.

In the quantum harmonic oscillator, this Gaussian term "$e^{\frac{-r^2}{2}}$" relates to the way the probability amplitude decays as a function of position; an exponential function that is characterized by the bell-shaped curve. These probabilities should properly decay to zero when "$x$" goes too far from the origin; that is, the probability of finding a particle far away from the equilibrium position goes to zero exponentially. The Gaussian shape reflects the property that in a quantum harmonic oscillator, due to the lowering of potential energies, one is most likely to find a particle near the equilibrium position; as one moves further from that point, in accordance with classical physics, this likelihood becomes lessened.

$$y = \hbar\omega \left(n + \frac{1}{2}\right)$$

The energy levels of the quantum harmonic oscillator are given by the expression above, where "$h$" is Reduced Planck's constant, "$\omega$" is the angular frequency of the oscillator, and "$n$" is the quantum number which are integer values, from 0.

$$K = \frac{mw}{h}$$

The equation above provides the Force constant, in which "$m$" is the mass of the body, "$\omega$" is the angular frequency, and "$\hbar$" is Planck's constant.

$$V(x) = \frac{1}{2} K x^2$$

This provides the potential energy function for a classical harmonic oscillator, where "$K$" is the force constant and "$x$" is the displacement.

$$k = m\omega^2$$

This is the spring constant in the context of the harmonic oscillator, defined as the product of mass "$m$", and the square of the angular frequency "$\omega$".

$$r = x\left(\sqrt{k}\right)$$

"$r$" is a dimensionless variable that scales the position "$x$" by a factor related to the spring constant "$k$". This substitution simplifies the wavefunction's argument and makes it dimensionless, which allows us to find underlying symmetries, and reduce the calculation's complexity.

## X. Mathematical Proof by Induction Of The Eigenstates Equation

In quantum mechanics, the oscillator becomes the quintessential model, making some useful insights concerning the nature of quantized systems. In this way, the energy spectrum of the quantum harmonic oscillator forms discrete and equidistantly spaced levels, which generally introduce the transition from the classical to the quantum description of physical systems. The conceptual leap from continuous energy spectra to discrete levels underpins much of modern quantum theory and iterates just how non intuitive quantum states are. In this regard, one is tasked with rigorously developing a summation that encapsulates the incremental progression of these energy levels, fundamentally characterizing the oscillator's quantum behaviour. We can take this logic further in a mathematical sense by systematically varying the relationship between successive energy levels. The following proof uses induction to confirm the general form of this summation and thus solidify the theoretical framework which describes the energy levels of the quantum harmonic oscillator;

$$\hbar\omega\left(a^{\dagger}a + \frac{1}{2}\right)\Psi = E\Psi \Rightarrow \hbar\omega\left(n + \frac{1}{2}\right)\Psi = E\Psi$$

*Summation of this equation from $P_n$ to $P_{k+1}$ is shown below.*

$u_n = \hbar\omega\left(n + \frac{1}{2}\right), u_0 = \frac{\hbar\omega}{2}, u_1 = \frac{3\hbar\omega}{2}, u_2 = \frac{5\hbar\omega}{2}, u_3 = \frac{7\hbar\omega}{2}$

$u_n = u_1 + (n-1)d = \dfrac{\hbar\omega}{2} + (n-1)\hbar\omega = \hbar\omega n - \dfrac{\hbar\omega}{2}$

$S_n = \dfrac{n}{2}[2u_1 + (n-1)d] = \dfrac{n}{2}[\hbar\omega + (n-1)\hbar\omega] = \dfrac{\hbar\omega n^2}{2}$

From this it can be shown that $P_n$ is:

$$\sum_{n=1}^{n}\left(\hbar\omega n - \frac{\hbar\omega}{2}\right) = \frac{\hbar\omega}{2} + \frac{3\hbar\omega}{2} + \frac{5\hbar\omega}{2} + \cdots + \hbar\omega n - \frac{\hbar\omega}{2} = \frac{\hbar\omega n^2}{2}$$

**Proof:** (By the principle of mathematical induction)

Let $n = 1$,

$$u_1 = \hbar\omega(1) - \frac{\hbar\omega}{2} = \frac{\hbar\omega}{2}$$

$$S_n = \frac{\hbar\omega(1)^2}{2} = \frac{\hbar\omega}{2}$$

If $P_k$ is true for $n \in \mathbb{Z}^+$, then

$$\sum_{k=1}^{k}\left(\hbar\omega k - \frac{\hbar\omega}{2}\right) = \frac{\hbar\omega}{2} + \frac{3\hbar\omega}{2} + \frac{5\hbar\omega}{2} + \cdots + \hbar\omega k - \frac{\hbar\omega}{2} = \frac{\hbar\omega k^2}{2}$$

Let "$k$" be "$k+1$",

$$\sum_{k=1}^{k}\left(\frac{\hbar\omega(1+2k)}{2}\right)=\frac{3\hbar\omega}{2}+\frac{5\hbar\omega}{2}+\frac{7\hbar\omega}{2}+\cdots+\frac{\hbar\omega(1+2k)}{2}=\frac{\hbar\omega(k+1)^2}{2}$$

Thus,

$$\begin{aligned}&\frac{\hbar\omega}{2}+\frac{3\hbar\omega}{2}+\frac{5\hbar\omega}{2}+\cdots+\frac{\hbar\omega(1+2k)}{2}\\&=\frac{\hbar\omega k^2}{2}+\frac{\hbar\omega(1+2k)}{2}\\&=\frac{\hbar\omega+2\hbar\omega k+k^2}{2}\\&=\frac{\hbar\omega(1+2k+k^2)}{2}\\&=\frac{\hbar\omega(k+1)^2}{2}\end{aligned}$$

Since $P_1$ is true, and $P_k$ is true whenever $P_{k+1}$ is true, $P_n$ is true for all $n \in \mathbb{Z}, n \geq 0$

## XI. Conclusion and Further Scope of Investigation

In conclusion, this research paper proves, using mathematical induction, the eigenstates equation, an understanding of which was built upon the foundational concepts including but not limited to wavefunctions, eigenvectors, and eigenvalues. We have gained new insight on the field, instrumental in ushering in all dependent developments in the form of quantum computing, nanotechnology, and beyond. With a stronger theoretical foundation, the development of quantum technologies shall revolutionise industries from cryptography and communication to medicine and artificial intelligence.

Though we are limited by our foundational understanding of the matter and the fact that the scientific community itself has only uncovered the tip of the iceberg that is quantum physics. Our work has provided a rigorous framework for understanding how quantum systems behave in specific, quantized states, as defined by their eigenfunctions, advancing our comprehension of quantum mechanics at a deeper level. Furthermore, the resources and tools needed to conduct quantum mechanical experiments are unavailable to us. This paper has been developed under the pressure of time constraints and educational burdens and though we are grateful for the support of our mentor, there is a relatively small community to guide us. Many of the abstract and complex concepts have proven increasingly arduous to grasp.

Given the tools, these concepts could be extended to more complex systems, we can address key challenges in modern physics and engineering. A more detailed study could be prepared given more time, resources and a deeper understanding of the matter which can be attained only after years of specialised study. Quantum mechanical experiments could bolster our knowledge

Nanotechnology can greatly benefit from quantum mechanics, enabling the creation of efficient, miniaturized devices with precise control over matter at the atomic level. This could lead to breakthroughs in energy storage and material science. This research not only strengthens our grasp of fundamental quantum mechanics but also lays the groundwork for the technologies that will define the coming decades. These studies are intertwined with nature itself, as seen when studying quantum thermodynamics, wherein standard laws of thermodynamics tend to fall apart. Through continued exploration and development in these domains, the potential for progress is unimaginable. As quantum technologies evolve, they will undoubtedly transform the way we understand and interact with the world even at its infinitesimal levels.

***Research Scholar**
****Research Scholar**
*****Department of Science**
******Scottish High International School**

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